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NOTE ON THE USE OF GROUP THEORY IN ELEMENTARY TRIGONOMETRY

By G. A. Miller

1. The applications of group theory in the development of some of the fundamental parts of trigonometry have received considerable attention ;* while some of these admit of interesting amplifications yet most of them relate to spherical trigonometry and are too complex to be immediately useful to the average teacher of elementary trigonometry. The present note relates to developments which are generally treated near the beginning of a course in trigonometry. While the mode of treatment which is outlined may not be suitable for the beginner, it will doubtless enable the teacher to see many points in a new light and to observe relations which had escaped notice.

The only group properties which are employed are those which belong to the cyclic group and the dihedral rotation group. While a knowledge of these groups will make the reading easier, yet it is not essential, in view of the elementary character of the considerations. In fact, these trigonometric developments furnish one of the easiest means of acquiring a clear notion of the groups in question. This paper has very close contact with the writer's paper entitled "A new chapter in trigonometry"† and includes the results developed in the first part of that paper.

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* E. Study, *Abhandlungen der Sächsischen Gesellschaft der Wissenschaften*, vol. 20 (1893), p. 83; F. Meyer, *Crelle*, vol. 115 (1895), p. 209; G. Chisholm, *Dissertation*. Göttingen, 1895; O. Pund, *Hamburger math. Gesellschaft*, vol. 3 (1897), p. 290; G. A. Miller, *Quarterly Journal of Mathematics*, vol. 37 (1906), p. 226.

†*Quarterly Journal of Mathematics*, loc. cit.

We assume as known that

$$\sin(90^\circ - x) = \cos x, \qquad \cos(90^\circ - x) = \sin x,$$
$$\sin(45^\circ - x) = \tfrac{1}{2}\sqrt{2}(\cos x - \sin x), \quad \cos(45^\circ - x) = \tfrac{1}{2}\sqrt{2}(\cos x + \sin x).$$

It is proposed to derive from these formulas, with the help of some elementary group theory, expressions for the functions of fourteen other angles in terms of those of x.

The operation of deriving the angle $45^\circ - x$ from x is of order 2, since $45^\circ - (45^\circ - x) = x$. Representing this operation by t and that of taking the complement of any angle by c, we have $t^2 = c^2 = 1$. The group $\{t, c\}$ generated by t and c is therefore of the dihedral rotation type and its order is twice the order of tc.* As tc signifies that an angle is subtracted from 45° and the complement of the resulting angle is obtained, it follows that $tc\,x = x + 45^\circ$, and therefore tc is of order 8. Hence $\{t, c\}$ is of order 16.

If $tc = t_1$, the operations of this group may be written as follows:

1	t_1^2	t_1^4	t_1^6	t_1	t_1^3	t_1^5	t_1^7
c	t_1^2c	t_1^4c	t_1^6c	t_1c	t_1^3c	t_1^5c	t_1^7c

All the operations of the second row are of order 2 while those of the first constitute the cyclic group of order 8. The operations of the first four columns constitute the group of order 8 known as the octic group or the group of the square. This subgroup is generated by c and $t_1c = s$, where s represents the operation of taking the supplement of an angle. If we apply these 16 operations to x the resulting angles in order (since angles are taken modulo 2π) are as follows:

x	$x + 90^\circ$	$x + 180^\circ$	$x + 270^\circ$	$x + 45^\circ$	$x + 135^\circ$	$x + 225^\circ$	$x + 315^\circ$
$90^\circ - x$	$-x$	$270^\circ - x$	$180^\circ - x$	$45^\circ - x$	$315^\circ - x$	$225^\circ - x$	$135^\circ - x$

If these 16 operations were applied to any other one of these angles the same 16 angles would result (in different arrangement), since the sixteen operations constitute a group.

* *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 424.

2. The trigonometric functions of sixteen angles. What precedes has been preliminary to expressing the trigonometric functions of each of these sixteen angles in terms of those of x assuming only that the equations giving the functions of $45° - x$ and of $90° - x$ in terms of those of x are known. It will be sufficient to do this for the sine and the cosine since the other functions are readily derived from these two. We begin with $x + 90°$ and for clearness give all the details in this instance.

$$\begin{aligned}\sin(x + 90°) &= \sin t_1^2 x = \sin tc\, tcx = \cos tctx = \tfrac{1}{2}\sqrt{2}(\cos tcx + \sin tcx)\\ &= \tfrac{1}{2}\sqrt{2}(\sin tx + \cos tx) = \tfrac{1}{2}(\cos x - \sin x + \cos x + \sin x) = \cos x,\\ \cos(x + 90°) &= \cos t_1^2 x = \sin tctx = \tfrac{1}{2}\sqrt{2}(\cos tcx = \sin tcx) = \tfrac{1}{2}\sqrt{2}(\sin tx - \cos tx)\\ &= \tfrac{1}{2}(\cos x - \sin x - \cos x - \sin x) = -\sin x.\end{aligned}$$

The functions of the angles corresponding to t_1^1 and t_1^3 may be directly derived from the preceding results as follows:

$$\begin{aligned}\sin(x + 180°) &= \sin t_1^4 x = \sin t_1^2 t_1^2 x = \quad \cos t_1^2 x = -\sin x,\\ \cos(x + 180°) &= \cos t_1^4 x = \cos t_1^2 t_1^2 x = -\sin t_1^2 x = -\cos x;\\ \sin(x + 270°) &= \sin t_1^3 x = \sin t_1^2 t_1^1 x = -\sin t_1^2 x = -\cos x,\\ \cos(x + 270°) &= \cos t_1^3 x = \cos t_1^1 t_1^2 x = -\sin t_1^1 x = \quad \sin x.\end{aligned}$$

The functions of the four angles corresponding to c, $t_1^2 c$, $t_1^1 c$, $t_1^3 c$ can be directly written down by means of our knowledge of the effect of the fundamental operation c and the results which have just been obtained. The work may be done as follows:

$$\begin{aligned}\sin(90° - x) &= \sin cx \quad = \cos x,\\ \cos(90° - x) &= \cos cx \quad = \sin x;\\ \sin(-x) &= \sin t_1^2 cx = \cos t_1^2 x = -\sin x,\\ \cos(-x) &= \cos t_1^2 cx = \sin t_1^2 x = \quad \cos x;\\ \sin(270° - x) &= \sin t_1^1 cx = \cos t_1^1 x = -\cos x,\\ \cos(270° - x) &= \cos t_1^1 cx = \sin t_1^1 x = -\sin x;\\ \sin(180° - x) &= \sin t_1^3 cx = \cos t_1^3 x = \quad \sin x,\\ \cos(180° - x) &= \cos t_1^3 cx = \sin t_1^3 x = -\cos x.\end{aligned}$$

The functions of the three angles corresponding to t_1^6, t_1^5, t_1^7 may be expressed as follows:

$$\begin{aligned}
&\sin(x+135^\circ)=\sin t_1^6 x=\sin t_1 t_1^6 x= \quad \cos t_1 x= \quad \sin tx= \quad \tfrac{1}{2}\sqrt{2}(\cos x-\sin x),\\
&\cos(x+135^\circ)=\cos t_1^6 x=\cos t_1 t_1^6 x=-\sin t_1 x=-\cos tx=-\tfrac{1}{2}\sqrt{2}(\cos x+\sin x);\\
&\sin(x+225^\circ)=\sin t_1^5 x=\sin t_1 t_1^5 x=-\sin t_1 x=-\cos tx=-\tfrac{1}{2}\sqrt{2}(\cos x+\sin x),\\
&\cos(x+225^\circ)=\cos t_1^5 x=\cos t_1 t_1^5 x=-\cos t_1 x=-\sin tx= \quad \tfrac{1}{2}\sqrt{2}(\sin x-\cos x);\\
&\sin(x+315^\circ)=\sin t_1^7 x=\sin t_1 t_1^7 x=-\cos t_1 x=-\sin tx= \quad \tfrac{1}{2}\sqrt{2}(\sin x-\cos x),\\
&\cos(x+315^\circ)=\cos t_1^7 x=\cos t_1 t_1^7 x= \quad \sin t_1 x= \quad \cos tx= \quad \tfrac{1}{2}\sqrt{2}(\cos x+\sin x).
\end{aligned}$$

As the functions of the angles which correspond to the remaining operations of our group may be obtained by first passing to the co-function and then proceeding as above it seems unnecessary to give them here. The thing which we desire to emphasize is the *method* of deriving these functions and that a knowledge of this group of order 16 gives us a clear insight into the number of possible similar ways of attaining the same results. Since this group is generated by 47 other pairs of operations we could have chosen the two fundamental angles in 48 distinct ways* and the derivation of the trigonometric functions of the remaining fourteen angles could have proceeded along the same lines. This clear comprehension of the various roads which lead to the same goal combined with the freedom to choose from such a variety of almost equally good roads constitutes one of the greatest sources of pleasure to the appreciative traveler in a mathematical field.

If it had been desired to find only the functions of the angles corresponding to the octic group we could have used the angles which correspond to two of its generators as fundamental instead of those corresponding to c and t. This was done in the article published in the *Quarterly Journal* to which we have already referred. The method pursued here presents the matter under a more comprehensive aspect and exhibits more extensive relations. There is however no limit to the generalizations along this line. For instance, if the operation t were replaced by an operation r, where rx indicates the result of subtracting the angle x from 30°: $rx = 30^\circ - x$, we should have to deal with a dihedral rotation group of order 24 which would again contain the octic group

* The number of ways of choosing two generators of the dihedral rotation group of order g is $\frac{1}{4}g\,\phi\,(\frac{1}{2}g)$, $\phi(n)$ being the totient, or indicator, of n.

as a subgroup. The considerations with respect to this group are practically the same as those given above.

3. The trigonometric functions of forty-eight angles. Since $rtx = x + 15°$, its order is 24 and $\{r, t\}$ is the dihedral rotation group of order 48. The angles which correspond to this group are x increased by the twenty-four multiples of 15° together with their negatives. The functions of these 48 angles can therefore be directly expressed in terms of those of x provided we know those of the angles corresponding to generators of $\{r, t\}$. As we have assumed a knowledge of those which correspond to t it is only necessary to add those which correspond to r; i. e.

$$\sin(30° - x) = \tfrac{1}{2}(\cos x - \sqrt{3}\sin x), \qquad \cos(30° - x) = \tfrac{1}{2}(\sqrt{3}\cos x + \sin x).$$

Hence,

$$\begin{aligned}
\sin(x + 15°) &= \sin rtx = \tfrac{1}{2}\sqrt{2}\,(\cos rx - \sin rx)\\
&= \tfrac{1}{4}\sqrt{2}(\sqrt{3}\cos x + \sin x - \cos x + \sqrt{3}\sin x)\\
&= \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\sin x + (\sqrt{3}-1)\cos x\},\\
\cos(x+15°) &= \cos rtx = \tfrac{1}{2}\sqrt{2}\,(\cos rx + \sin rx)\\
&= \tfrac{1}{4}\sqrt{2}(\sqrt{3}\cos x + \sin x + \cos x - \sqrt{3}\sin x)\\
&= \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\cos x - (\sqrt{3}-1)\sin x\}.
\end{aligned}$$

The functions of the remaining 47 angles may be readily found from these results if we observe that $\{r, t\} \equiv \{rt, c\}$, and hence $rt = t_2$ and c may be used as the fundamental operations. To obtain the functions of $75° - x = rtcx$ it is only necessary to observe that $\sin rtcx = \cos rtx$ and $\cos rtcx = \sin rtx$, from which it follows that

$$\sin(75° - x) = \sin t_2cx = \cos t_2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\cos x - (\sqrt{3}-1)\sin x\},$$

$$\cos(75° - x) = \cos t_2cx = \sin t_2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\sin x + (\sqrt{3}-1)\cos x\}.$$

Further,

$$\begin{aligned}
&\sin(x + 30°) = \sin t_2^2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\sin t_2x + (\sqrt{3}-1)\cos t_2x\}\\
&= \tfrac{1}{8}\{(4+2\sqrt{3})\sin x + 2\cos x + 2\cos x - (4-2\sqrt{3})\sin x\} = \tfrac{1}{2}(\cos x + \sqrt{3}\sin x),\\
&\cos(x + 30°) = \cos t_2^2x = \tfrac{1}{4}\sqrt{2}\{(\sqrt{3}+1)\cos t_2x - (\sqrt{3}-1)\sin t_2x\}\\
&= \tfrac{1}{8}\{(4+2\sqrt{3})\cos x - 2\sin x - 2\sin x - (4-2\sqrt{3})\cos x\} = \tfrac{1}{2}(\sqrt{3}\cos x - \sin x);
\end{aligned}$$

$$\sin(x+60°) = \sin t_2^1 x = \tfrac{1}{2}(\cos t_2^6 x + \sqrt{3}\sin t_2^6 x)$$
$$= \tfrac{1}{4}(\sqrt{3}\cos x - \sin x + \sqrt{3}\cos x + 3\sin x) = \tfrac{1}{2}\sqrt{3}\cos x + \sin x),$$
$$\cos(x+60°) = \cos t_2^1 x = \tfrac{1}{2}(\sqrt{3}\cos t_2^6 x - \sin t_2^6 x)$$
$$= \tfrac{1}{4}(3\cos x - \sqrt{3}\sin x - \cos x - \sqrt{3}\sin x) = \tfrac{1}{2}(\cos x - \sqrt{3}\sin x).$$

The preceding examples appear ample to exhibit the method of work and the numerous relations which it sets forth. The two generating operations of $\{r, t\}$ could be selected in $36 \cdot 8 = 288$ different ways so that there is an almost endless variety of ways from which to choose in deriving the functions of these 48 angles along this line. The method aids in deducing a great deal from a few known facts but is of no value without such knowledge. It has a theoretic and clarifying influence but does not seem to fit in a brief practical course in trigonometry. If the trigonometric functions of a series of angles is known, those of any multiple (positive or negative) of the highest common factor of all the possible differences between these angles may be found in this way.* In fact, we are dealing simply with the group of subtraction from several fixed numbers with respect to modulus 2π, which is considered in the last article to which we have referred. The present article may serve as an illustration of the utility of such groups.

* ANNALS OF MATHEMATICS, ser. 2, vol. 6 (1905), p. 41.

THE ASYMPTOTIC LINES ON THE ANCHOR RING

By Marion Ballantyne White

The purpose of this paper is to determine the equations of the asymptotic lines on the surface of the anchor ring when its equations are expressed in parameter form and, with the aid of elliptic functions, to examine their geometrical properties.* The first three sections are devoted to the equations of asymptotic lines in general, to their applications to the anchor ring, and to the expression of the equations of the asymptotic lines in terms of the elliptic functions of Jacoby. After determining the general form of the asymptotic curves, the anchor rings for which they are closed will be discussed. The paper concludes with the computation of a numerical example.

1. Asymptotic Lines in General. Consider a surface whose equations are

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

and a plane tangent to that surface at a point M whose coordinates are (u, v). The distance of any point $(u + h, v + k)$ on the surface in the neighborhood of M from this tangent plane is represented in Bianchi's notation† by

$$\delta = \tfrac{1}{2}(Dh^2 + 2D'hk + D''k^2) + \eta, \tag{1}$$

* The only literature I have found on this subject is a brief summary of an article by T. Matoda, "Asymptotic lines on a circular ring," *Tokio Math. Soc.*, vol. 4, pp. 217–219, in the *Jahrbuch über die Fortschritte der Mathematik*. vol. 21, p. 765, and also an article by Prof. Maschke, "Asymptotic lines on a circular ring," *Bull. Amer. Math. Soc.*, vol. 2, 1895, pp. 19–21, in which the equations of the lines are briefly reduced to a form involving Weierstrassian elliptic functions without any corresponding geometrical investigation.

† L. Bianchi, *Vorlesungen über Differentialgeometrie*, translation by M. Lukat (1896), pp. 61, 87, 104.

where η is an infinitesimal of the third order in h and k. The expressions for D, D' and D'' are:

$$(2)\qquad D=\frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

$$D'=\frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

$$D''=\frac{1}{\sqrt{EG-F^2}}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},$$

where x_u, x_v are the partial derivatives of x with respect to u and v, and as usual

$$(3)\qquad E=x_u^2+y_u^2+z_u^2,\quad F=x_u x_v+y_u y_v+z_u z_v,\quad G=x_v^2+y_v^2+z_v^2.$$

The surface in the neighborhood of M lies on one side of, or is cut by, the tangent plane according as δ retains or changes its sign. This δ will always be positive for sufficiently small values of h and k if the two factors of $Dh^2+2D'hk+D''k^2$ are conjugate imaginary factors, i. e., if $DD''-D'^2>0$; in this case the surface is said to have elliptic curvature at the point M. If $DD''-D'^2<0$, δ may take positive or negative values for small values of h and k and the curvature at M is said to be hyperbolic. Parabolic curvature occurs at points where $DD''-D'^2=0$.

The curve in the uv-plane corresponding to the equation

$$(4)\qquad \delta=\tfrac{1}{2}(Dh^2+2D'hk+D''k^2),$$

where δ is constant and η has been neglected, is approximately the image in the uv-plane of the section of the surface by a plane parallel to, and at a distance δ from the tangent plane.

Two systems of curves on a surface are conjugate systems if the directions of their tangents at every point on the surface correspond to directions in the uv-plane which are conjugate with respect to the conic (4). From the theory of conic sections, the condition for this is*

* See also: Bianchi, *Vorlesungen über Differentialgeometrie*, p. 108.

$$(5) \qquad D du\, \delta u + D'(du\, \delta v + dv\, \delta u) + D'' dv\, \delta v = 0,$$

where d and δ are used to distinguish between the increments along the two curves. In particular, *if a curve on the surface is such that the direction of the tangent at any point is conjugate to itself, the curve is called an asymptotic line on the surface.* It follows immediately from the last equation that the differential equation for the asymptotic lines is

$$D du^2 + 2D' du\, dv + D'' dv^2 = 0.$$

As this is a quadratic in du and dv, the asymptotic lines evidently form a double system. If they are real, the equation must have real roots. It follows therefore that the real asymptotic lines lie only on that part of the surface where $DD'' - D'^2 < 0$.

To summarize: *The asymptotic lines on the surface* $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ *form a double system of curves whose differential equation is*

$$(6) \qquad D du^2 + 2D' du\, dv + D'' dv^2 = 0,$$

D, D' and D'' having the values given in equations (2). *They are real only on that part of the surface where* $DD'' - D'^2 < 0$.

2. The Equations of the Asymptotic Lines on the Anchor Ring. Let us now apply the general formulas just obtained to the particular case of the anchor ring. The anchor ring to be considered is generated by the revolution of a circle with radius b and center at $(a, 0, 0)$ about the

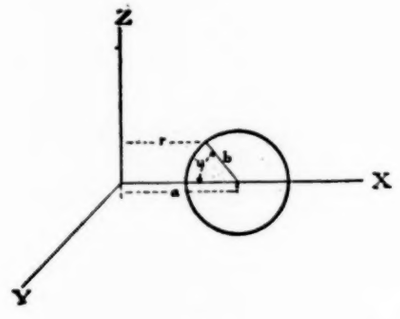

FIG. 1.

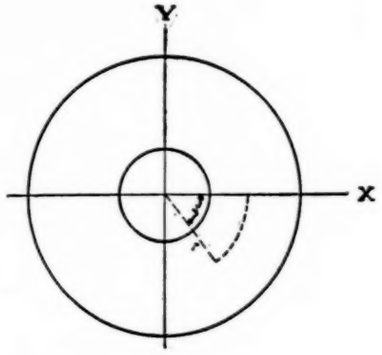

FIG. 2.

z-axis. Instead of using the equation in the form $F(x, y, z) = 0$, we may express x, y and z as functions of two parameters r and v or of u and v, whose meaning is indicated in the accompanying figures.

By referring to these figures, it is easy to see that the equations of the anchor ring in terms of u, v are:

$$x = (a - b\cos u)\cos v, \tag{7}$$

$$y = (a - b\cos u)\sin v,$$

$$z = b\sin u,$$

the parameter lines being the meridians and parallels on the surface.

By means of the formulas (2) and (3), E, F, G, D, D' and D'' may be calculated, with the following results:–

$$E = b^2,\ F = 0,\ G = (a - b\cos u)^2,\ \sqrt{EG - F^2} = b(a - b\cos u),$$

$$D = -b,\ D' = 0,\qquad D'' = (a - b\cos u)\cos u.$$

It is evident that F must equal zero, since x_u, y_u, z_u and x_v, y_v, z_v are the direction cosines of the parameter lines, which intersect orthogonally.

Substituting these values of D, D', and D'' in the equation (6) of the asymptotic lines, it becomes

$$-b\,du^2 + (a - b\cos u)\cos u\,dv^2 = 0, \tag{8}$$

from which we derive

$$dv = \frac{\pm\sqrt{b}}{\sqrt{\cos u(a - b\cos u)}}\,du, \tag{9}$$

or

$$v = \pm\sqrt{b}\int_0^u \frac{du}{\sqrt{\cos u\,(a - b\cos u)}} + C. \tag{10}$$

The constant C may be determined so that the asymptotic line passes through an arbitrary point (u_0, v_0) on the ring, provided (u_0, v_0) lies in the portion of the surface where $DD'' - D'^2 < 0$. Then

$$C = v_0 \mp \sqrt{b}\int_0^{u_0} \frac{du}{\sqrt{\cos u(a - b\cos u)}}, \tag{11}$$

and

$$v = \pm\sqrt{b}\int_0^u \frac{du}{\sqrt{\cos u(a - b\cos u)}} + v_0 \mp \sqrt{b}\int_0^{u_0} \frac{du}{\sqrt{\cos u(a - b\cos u)}}. \tag{12}$$

By a rotation about the z-axis, the equation for any asymptotic line can therefore be finally reduced to one of the forms

$$v = \pm \sqrt{b} \int_0^u \frac{du}{\sqrt{\cos u (a - b \cos u)}}. \tag{13}$$

In §1 we found that these lines can be real only when $DD'' - D'^2 = -b \cos u\ (a - b \cos u) < 0$. Since $a > b$, we have $a - b \cos u > 0$ and therefore this inequality holds only when $-\frac{\pi}{2} < u < \frac{\pi}{2}$. These are exactly the values for which the denominator in the integral of (13) is real.

Our results are then as follows:

1) *The equations of any asymptotic lines on the anchor ring may, by a rotation about the z-axis, be reduced to one of the forms* (13).

2) *The asymptotic lines are real only on the inner half of the ring, that is, in the region* $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

3. The Equations in terms of Elliptic Functions. The purpose of the present section is to express the equations of the asymptotic lines in terms of the elliptic functions. That the integral (13) is really an elliptic integral, appears when u is expressed in terms of $r = a - b \cos u$ (see figure 1). Since $\cos u = (a - r)/b$, the integral (13) becomes

$$v = \pm b \int_{a-b}^{r} \frac{dr}{\sqrt{r(a - r)\left[(a - r)^2 - b^2\right]}}. \tag{14}$$

Hence r, and therefore also u, are elliptic functions of v.

To reduce this integral to the normal form, Durège's method* will be used. The integral to be reduced is in the form $\int_{r_1}^{r_2} \frac{dr}{\sqrt{AR^2}}$, where $A = 1$ and the four roots of $R^2 = 0$ are 0, $a - b$, a, and $a + b$. We will now introduce a new variable ξ, so chosen that the values of ξ which correspond to 0, $a - b$, a, and $a + b$ are $-1/k$, -1, 1, and $1/k$ respectively. If the four roots of the

* H. Durège, *Theorie der elliptischen Functionen* (1887), p. 34.

equation in r are designated by π, p, q, and κ, we have the following correspondence between the values of ξ and r.

$$(15) \qquad \begin{cases} \xi = 1/k, & 1, & -1, & -1/k, \\ r = & \pi, & p, & q, & \kappa, \end{cases}$$

where $\pi = a + b, \quad p = a, \quad q = a - b, \quad \kappa = 0.$

The following formulas are the ones given by Durège to effect the transformation to the normal form when the roots of $R^2 = 0$ are all real. The relation between r and ξ is given by*

$$(16) \qquad \frac{r-p}{r-q} = P\frac{1-\xi}{1+\xi},$$

where

$$(17) \qquad P = \pm\sqrt{\frac{(p-\pi)(p-\kappa)}{(q-\pi)(q-\kappa)}}.$$

The value of k is determined from

$$(18) \qquad \frac{1-k}{1+k} = +\sqrt{\frac{(p-\pi)(q-\kappa)}{(q-\pi)(p-\kappa)}}.$$

The values of the old and new integrals are related by the equation

$$(19) \qquad \frac{dr}{\sqrt{A(r-p)(r-q)(r-\pi)(r-\kappa)}} = \frac{2\sqrt{k}}{\sqrt{A(p-q)(\pi-\kappa)}} \cdot \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}},$$

or

$$(20) \qquad \frac{dr}{\sqrt{A(r-p)(r-q)(r-\pi)(r-k)}} = \frac{2\sqrt{k}}{\sqrt{A(p-q)(\pi-\kappa)}} \cdot \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},$$

where

$$(21) \qquad \xi = \sin\phi.$$

On making the substitutions for the anchor ring from (15) in (16), (17), (18), and (20), we obtain the following results:

$$(22) \qquad r = a + \frac{bP(1-\xi)}{1+\xi-P(1-\xi)},$$

$$(23) \qquad P = -\sqrt{\frac{a}{2(a-b)}},$$

* Durège, loc. cit., p. 44, ff.

the minus sign being chosen, since it is desired that r and ξ increase simultaneously. Also

$$k = \frac{\sqrt{2a} - \sqrt{a-b}}{\sqrt{2a} + \sqrt{a-b}}, \tag{24}$$

and

$$v = \pm 2\sqrt{\frac{bk}{a+b}} \int_{-1}^{\xi} \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} = \pm 2\sqrt{\frac{bk}{a+b}} \int_{-\frac{\pi}{2}}^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}. \tag{25}$$

Let

$$2\sqrt{\frac{bk}{a+b}} = \frac{1}{a}. \tag{26}$$

Then in the notation of Durège,* equation (25) may be written

$$v = \pm \frac{1}{a} \int_{-\frac{\pi}{2}}^{\phi} \frac{d\phi}{\Delta\phi}, \tag{27}$$

or

$$\pm av = \int_0^{\phi} \frac{d\phi}{\Delta\phi} + \int_{-\frac{\pi}{2}}^{0} \frac{d\phi}{\Delta\phi} = \int_0^{\phi} \frac{d\phi}{\Delta\phi} + K, \tag{28}$$

or, solving for ϕ,

$$\phi = \operatorname{am}(\pm av - K). \tag{29}$$

The formula for calculating $\xi = \sin\phi$ is†

$$\operatorname{sn}(u-v) = \frac{\operatorname{sn}u \operatorname{cn}v \operatorname{dn}v - \operatorname{sn}v \operatorname{cn}u \operatorname{dn}u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

u, v in this case having the values $\pm av$, K. After using the relation $\operatorname{cn} K = 0$ and $\operatorname{sn} K = 1$, this gives

$$\xi = -\frac{\operatorname{cn}(\pm av)\operatorname{dn}(\pm av)}{\operatorname{dn}^2(\pm av)} = -\frac{\operatorname{cn}av}{\operatorname{dn}av}, \tag{30}$$

since $\operatorname{cn}av$ and $\operatorname{dn}av$ are even functions.

It now remains to express u in terms of these elliptic functions. With the help of equation (22), since $\cos u = (a-r)/b$ we obtain

$$\cos u = \frac{-P(1-\xi)}{1+\xi+P(1-\xi)}. \tag{31}$$

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* Loc. cit., pp. 8, 17.

† Durège, loc. cit., pp. 106, 116.

Substituting (30) in (31) we find that *the equations defining u in terms of v, which determine the asymptotic lines on the anchor ring, may be expressed in the form:*

$$(32)\quad \begin{cases} (a) \quad \cos u = -\dfrac{P(\mathrm{dn}\,av + \mathrm{cn}\,av)}{\mathrm{dn}\,av - \mathrm{cn}\,av - P(\mathrm{dn}\,av + \mathrm{cn}\,av)}, \\ (b) \quad \sin u = \pm \dfrac{\sqrt{(\mathrm{dn}\,av - \mathrm{cn}\,av)^2 - 2P(\mathrm{dn}^2 av - \mathrm{cn}^2 av)}}{\mathrm{dn}\,av - \mathrm{cn}\,av - P(\mathrm{dn}\,av + \mathrm{cn}\,av)}, \end{cases}$$

where the values of P and a are to be taken from equations (23) *and* (26).

4. The Geometric Character of the Asymptotic Lines. It has already been shown that the asymptotic lines lie on the inner half of the anchor ring. Their form may be determined by examining equations (32) and (9) which give u and du/dv respectively, and also the equation

$$(33)\quad \frac{d^2u}{dv^2} = \frac{\sin u(2b\cos u - a)}{2b}$$

obtained by taking the v-derivative of (9).

In equation (32, b), the $\pm$ signs indicate that the asymptotic lines are related in pairs symmetrical with respect to the v-axis. Only the positive sign need be considered. The sign of (9) is not necessarily the same as that chosen in (32, b). The correspondence will be evident from the following discussion.

The positive function u defined by equations (32) is, in the first place, periodic in v since the elliptic functions have a period of $4K$, where K is defined by

$$(34)\quad K = \int_0^{\pi/2} \frac{d\phi}{\Delta\phi}.$$

The period in av is $4K$, that is

$$u\left(v + \frac{4K}{a}\right) = u(v).$$

Therefore to determine the graph of the line, we need only consider u as a function of v in the interval $0 \leqq u \leqq 4K/a$. This interval can be still further restricted since

$$\mathrm{cn}(2K \pm w) = -\mathrm{cn}\,w, \quad \mathrm{dn}(2K \pm w) = \mathrm{dn}\,w,$$

and therefore

$$u\left(\frac{2K}{a}+v\right)=u\left(\frac{2K}{a}-v\right).$$

The curve is thus symmetrical with respect to the ordinate $v=2K/a$, and it follows that its form is completely determined when the function u has been investigated in the interval $0 \leqq u \leqq 2K/a$.

Equations (32) and (9) show that u is a continuous function of v and has a continuous derivative. Moreover, du/dv can vanish only when $\cos u = 0$, that is when $\operatorname{cn} av + \operatorname{dn} av = 0$, and it can be shown that this equation is true in the interval $0 \leqq u \leqq 2K/a$, only for $av = 2K$. The limits between which u varies can be found by putting $v=0$ and $v=2K/a$ successively in equations (32). When $av=0$, since $\operatorname{dn}(0)=1$ and $\operatorname{cn}(0)=1$, it follows that $u=0$. For $av=2K$, we have $\operatorname{dn} av = 1$, $\operatorname{cn} av = -1$, and therefore $u=\pi/2$. It is evident that for the positive function u defined by equation (32), the sign in equation (9) must be the positive one, and that u constantly increases as v varies from 0 to $2K/a$.

The equation (33) shows that the curve is always concave toward the v-axis in the interval $0 \leqq v \leqq 2K/a$, unless $2b > a$. In any case there is an inflexion point at $u=0$, and when $2b > a$, there is one also at $u=\cos^{-1}(a/2b)$, at which the curvature changes from convex to concave toward the v-axis.

From the foregoing discussion it is evident that the images in the uv plane of the asymptotic lines have the properties shown in figure 3.

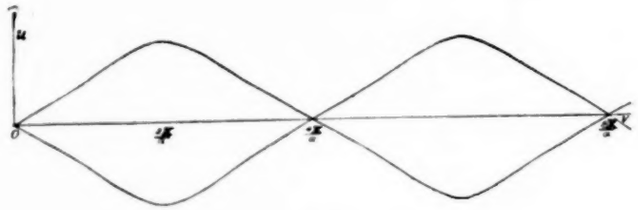

FIG. 3.

On the anchor ring (see figure 4) *the asymptotic lines oscillate back and forth across the inner equator between the two circles* $u=\pi/2$ *and* $u=-\pi/2$. *Any two asymptotic lines are similar and can be made to coincide by a rotation about the axis of the ring. A section of one of the lines between*

two successive contacts with the circle $u = -\pi/2$ *is symmetrical with respect to a plane through its point of contact with the circle* $u = \pi/2$ *and the axis of the ring, and any such section may be made to coincide with any other by a rotation about the same axis.*

The period of the positive function $u(v)$ defined by equations (32) was found to be $4K/a$. But if the positive values of u in the interval $0 < v < 4K/a$

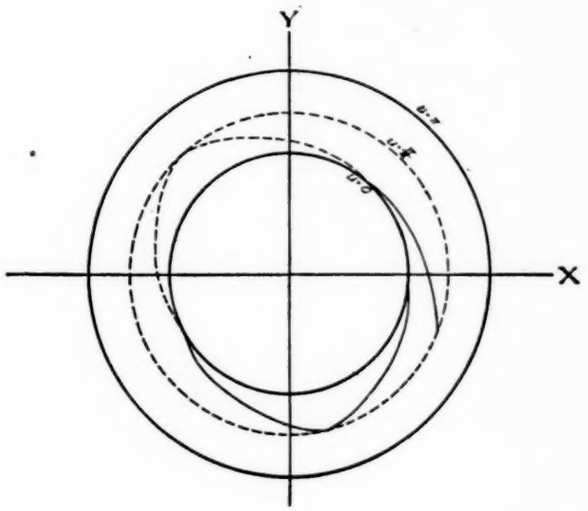

FIG. 4.

are taken with the negative values in $4K/a < v < 8K/a$, and so on as indicated in figure 3, so that the line defined has a continuously turning tangent, the resulting function of v will evidently have the period $8K/a$.

5. Anchor Rings with Closed Asymptotic Lines. The problem here is to determine what form the anchor ring must take if the asymptotic lines are to be closed curves. Since they have a period of $8K/a$, the condition to be satisfied is that some multiple of $8K/a$ must be a divisor of a multiple of 2π, or stating it algebraically,

$$\frac{8Km}{a} = 2\pi n,$$

where m and n are positive integers. From this equation, since K has the value (34), the condition becomes

$$\frac{\pi n}{4m} = \frac{1}{a}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}. \tag{35}$$

The right hand member depends on a and k and, as is evident from equations (24) and (26), may be considered a function of the ratio b/a. Our first conclusion then is:

The value of the ratio b/a determines whether or not the asymptotic lines are closed; i. e. the closure of the lines depends on the *proportions* and not on the *size* of the ring.

The problem now is to determine whether there is any fixed ratio between a and b which will make the right-hand member of (35) equal to the constant $\pi n/4m$. We will first determine a and k in terms of w where $w = b/a$. Using the values of k and a in (24) and (26). Then

$$k = \frac{\sqrt{2a} - \sqrt{a - b}}{\sqrt{2a} + \sqrt{a - b}} = \frac{\sqrt{2} - \sqrt{1 - w}}{\sqrt{2} + \sqrt{1 - w}}, \tag{36}$$

$$a = \tfrac{1}{2}\sqrt{\frac{a + b}{bk}} = \tfrac{1}{2}\sqrt{\frac{w + 1}{wk}}. \tag{37}$$

If we designate by $\phi(w)$, the function in the right-hand member of (35), the problem is to solve the equation,

$$\phi(w) = \frac{\pi n}{4m}.$$

For this purpose we will study the graph of $\phi(w)$, and since on the anchor ring $a > b$, we may confine ourselves to the interval in which $0 < w < 1$. For $w = 0$, $\phi(w) = 0$, and when $w = 1$, $\phi(w) = \infty$, since $k = 1$ and the integral in (35) is infinite. Therefore the curve passes through the origin and has $w = 1$ as an asymptote.

The derivative $\phi'(w)$ has the value

$$\phi'(w) = -\frac{1}{a^2}\cdot\frac{da}{dw}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} + \frac{1}{a}\int_0^{\pi/2}\frac{k\sin^2\phi d\phi}{(\sqrt{1 - k^2\sin^2\phi})^3}\cdot\frac{dk}{dw}. \tag{38}$$

From (36) we derive

$$\frac{dk}{dw} = \frac{\sqrt{2}}{\sqrt{1-w}\,(\sqrt{2}+\sqrt{1-w})^2}. \tag{39}$$

In (37) a is a function of both w and k; therefore

$$\frac{da}{dw} = \frac{\partial a}{\partial w} + \frac{\partial a}{\partial k}\cdot\frac{dk}{dw} \tag{40}$$

$$= -\frac{1}{4\sqrt{k}}\left[\frac{1}{\sqrt{w^3(1+w)}} + \sqrt{\frac{1+w}{w}}\cdot\frac{\sqrt{2}}{k\sqrt{1-w}\,(\sqrt{2}+\sqrt{1-w})^2}\right].$$

Substituting these values from (39) and (40) in equation (38), it appears that $\phi'(w)$ has the value

$$\frac{1}{4a^2\sqrt{k}}\left[\frac{1}{\sqrt{w^3(1+w)}} + \sqrt{\frac{1+w}{w}}\cdot\frac{\sqrt{2}}{k\sqrt{1-w}(\sqrt{2}+\sqrt{1-w})^2}\right]\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$+\frac{1}{a}\frac{\sqrt{2}}{\sqrt{1-w}(\sqrt{2}+\sqrt{1-w})^2}\int_0^{\pi/2}\frac{k\sin^2\phi\,d\phi}{(\sqrt{1-k^2\sin^2\phi})^3},$$

which is positive for $0 < w < 1$, and therefore $\phi(w)$ increases monotonically in that interval.

The general form of the graph of $\phi(w)$ is that of the curve (1) in figure 5. The straight line $\phi = \pi n/4m$ is represented by (2). Curves (1)

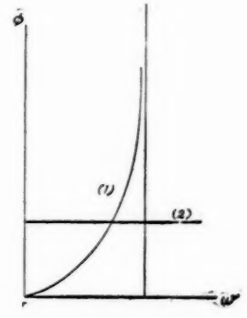

FIG. 5.

and (2) have but one point of intersection, which indicates that *for every value of m and n that may be chosen, there is a fixed ratio between a and b for which*

the asymptotic lines will be closed: i. e., the proportions for a ring may be so chosen that the asymptotic lines close after any arbitrary even number $2m$ of intersections with the inner equator, and an arbitrary number n of windings about the z-axis.

6. A Numerical Example. In this section it will be shown how the branch of the curve given in Figure 3, may be plotted from actual computation. The constants $a = 18$, $b = 14$ have been chosen so as to illustrate the case where there is a point of inflection for $\cos^{-1}(a/2b)$. The values $k = 0.5$ and $a = 4/\sqrt{14}$ are readily computed from equations (24) and (26).

To find K, Landen's first transformation* has been used. The object of this transformation is to increase the modulus k by successive steps, the upper limit of the integral in (34) being decreased incidentally until k is so near to unity that the integral has the approximate value

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1 - \sin^2\phi}} = \log\left[\tan\frac{\phi}{2} + \frac{\pi}{4}\right].$$

The first transformation takes K into

$$K = \frac{2}{1 + k}\int_0^{\phi_1} \frac{d\phi_1}{\sqrt{1 - k_1^2\sin^2\phi_1}},$$

where k_1 and ϕ_1 are determined by the equations

$$k_1 = \frac{2\sqrt{k}}{1 + k}, \qquad \sin(2\phi_1 - \phi) = k\sin\phi.$$

After two applications of the formula, k_2 is so near to unity that K may be written in the form

$$K = \sqrt{\frac{k_1}{k}}\log\tan\left(\frac{\phi_2}{2} + \frac{\pi}{4}\right),$$

where $\frac{\phi_2}{2} + \frac{\pi}{4} = 73^\circ\ 41'\ 1.2''$. Hence we find $K = 1.686$.

* W. E. Byerly, *Integral Calculus*, p. 215.

The period of the asymptotic lines was found to be $8K/a$, which in the present problem therefore becomes 12.61.

From the equations (32) and (9) of the asymptotic lines, values of u and $\frac{du}{dv}$ corresponding to special values of v may be computed as shown in the following table:

v	u	$\frac{du}{dv}$
0	0	0.524
$\frac{K}{a} = 1.576$	0.927	0.6415
$\frac{2K}{a} = 3.152$	$\pi/2$	0

There is a point of inflection when $u = 0.872$ radians.

To find other points (u, v) on the curve, values of the integral

$$v = \int_0^{\pi/2} \frac{\sqrt{7}}{\sqrt{\cos u(9 - 7\cos u)}}\, du,$$

obtained from equation (13), may be computed by means of the trapezoidal rule for computing areas under a curve.

This rule is:* "Add together the halves of the extreme ordinates and the whole of the intermediate ordinates, and multiply the result by the common interval." To apply the rule, the interval $\pi/2$ was divided into eighteen subdivisions of 0.087 each, the ordinate at each division point calculated and substituted in the formula just given. A table is appended which gives the resulting values of u and v in radians; by means of which the curve in figure 3, may be easily plotted for the interval $0 \leqq v \leqq 2K/a$.

* Williamson, *Integral Calculus*, p. 212.

For the projection of the asymptotic line on the xy plane, shown in

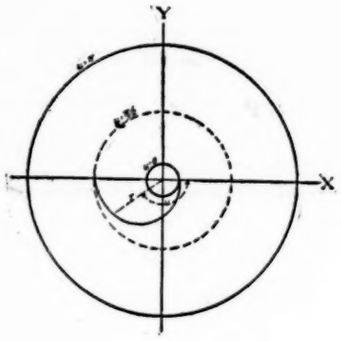

FIG. 6.

figure 6, the values for r were calculated from $r = a - b \cos u$, and these are given in the third column of the table.

TABLE

u	v	r
.000	.000	4.00
.087	.162	4.04
.174	.330	4.20
.262	.481	4.46
.349	.637	4.84
.436	.788	5.30
.524	.934	5.80
.611	1.070	6.30
.698	1.206	7.23
.785	1.350	8.10
.873	1.480	9.10
.960	1.620	9.80
1.047	1.762	11.00
1.134	1.804	12.10
1.228	2.050	13.20
1.309	2.203	14.30
1.396	2.396	15.50
1.483	2.631	16.70
1.570	3.152	18.00

UNIVERSITY OF WISCONSIN, AUGUST, 1906.

ON CERTAIN THEOREMS OF MEAN VALUE FOR ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

By D. R. Curtiss

On account of the fundamental importance of the theorems of mean value for real functions of a real variable an investigation of the validity of the corresponding formulas for analytic functions of a complex variable may prove of interest, even if the results obtained are of no great value in themselves.

Darboux has established theorems closely analogous to those expressed by the formulas:

$$(1) \qquad f(x) - f(a) = (x-a)\, f'[a + \theta\,(x-a)]$$

$$(2) \qquad \int_a^x f(y)\,\phi(y)\,dy = f[a + \theta\,(x-a)] \int_a^x \phi(y)\,dy$$

$$\Big\} \left(|\theta - \tfrac{1}{2}| < \tfrac{1}{2}\right)$$

but with the essential difference that in the second members of the above equations there appears a factor λ whose absolute value is less than 1.* So far as I have been able to ascertain, no investigation of (1), (2), and allied formulas for real functions of a real variable has been published which considers their validity without change of form when x, y, θ and a take on complex values, and f and ϕ are analytic functions of their arguments.

We may regard (1) and (2) as equations defining θ as an implicit function of x. When f and ϕ are real functions of a real variable subject to certain restrictions, at least one of the branches of this implicit function takes on a real value between 0 and 1 for each value of x in a given interval. The question arises whether equations (1) and (2) have still a solution θ of which at least one branch is always to be found in a certain limited region, when f and ϕ are functions of a complex variable analytic in a region B which contains the point a. We shall prove that there exists a neighborhood of a within which formulas (1) and (2), as well as others similar to these, remain true. In the case of real functions of a real variable something can be said as to the extent of this

* *Journal de Mathématiques*, ser. 3, vol. 2 (1876), pp. 291, 294. Cf. Stolz, *Differential- und Integralrechnung*, part II, pp. 92-95. A recent publication by Brunn — *Beziehungen des Du Bois-Reymondschen Mittelwertsatzes zur Ovaltheorie* — gives some interesting theorems which are not, however, entirely in terms of functions of a complex variable.

region; in the present paper, where only analytic functions of a complex variable are considered, no positive results of this kind have been obtained.

1. The Mean Value Theorem for Derivatives. The following theorem corresponds to formula (1):

THEOREM I. *Let a be an interior point of a region B throughout which $f(z)$ is analytic. Then there exists a neighborhood A of a such that for any value of z in A the equation*

$$F(\theta, z) \equiv f(z) - f(a) - (z-a)\, f'[a + \theta\,(z-a)] = 0$$

is satisfied by a value of θ which verifies the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

To prove this theorem we expand $F(\theta, z)$ in a series proceeding according to powers of $z - a$. We thus obtain a development

$$F(\theta, z) = \sum_{k=2}^{\infty} \left[\frac{1}{k!} - \frac{\theta^{k-1}}{(k-1)!}\right] f^{(k)}(a)(z-a)^k, \tag{3}$$

which converges for all values of z within a circle C whose center is a, provided θ satisfies the inequality $|\theta - \tfrac{1}{2}| < \tfrac{1}{2}$.

If $f(z)$ has the form $\alpha z + \beta$, the above theorem is evidently true since $F(\theta, z)$ then vanishes identically. In all other cases, at least one of the successive derivatives of $f(z)$, beginning with the second, will not vanish at a. Accordingly we have

$$f''(a) = f'''(a) = \cdots = f^{(n-1)}(a) = 0, \quad f^{(n)}(a) \neq 0.$$

We now make use of an auxiliary function $\Phi(\theta, z)$, defined as follows:

$$\begin{aligned} \Phi(\theta, z) &= \frac{F(\theta, z)}{(z-a)^n}, \quad z \neq a, \\ \Phi(\theta, a) &= \left(\frac{1}{n!} - \frac{\theta^{n-1}}{(n-1)!}\right) f^{(n)}(a). \end{aligned} \tag{4}$$

For this function we have

$$\Phi\left(\sqrt[n-1]{\frac{1}{n}}, a\right) = 0,$$

$$\frac{\partial}{\partial \theta}\Phi\left(\sqrt[n-1]{\frac{1}{n}}, a\right) \neq 0,$$

where $\sqrt[n-1]{\frac{1}{n}}$ is the real solution, lying between 0 and 1, of the equation

$$\theta^{n-1} = \frac{1}{n}.$$

Accordingly, by a well known existence theorem for implicit functions,* there is a neighborhood A of a such that for each point z of A the equation

$$\Phi(\theta, z) = 0$$

has a solution θ interior to the circle defined by the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

But for $z \neq a$, $F(\theta, z)$ vanishes in the same points as $\Phi(\theta, z)$, while $F(\theta, a)$ is identically zero. Our theorem is thus established.

This method of demonstration also shows that, if we except the case where $f(z)$ has the form $az + \beta$, each branch of the implicit function θ is analytic when z is in the neighborhood of a, and approaches one of the $(n-1)$th roots of $\frac{1}{n}$ when z approaches a.

2. Restrictions on Theorem I, and a Related Theorem. It should be noted carefully that we have proved only the *existence* of a region A in which the above theorem holds,—nothing has been said as to its size. In fact the determination of such a region for a given function and a given point a is usually a difficult problem. We can easily construct elementary functions such that, *no matter how small* $|b - a|$ *may be, when* b *has once been fixed there is no value of* θ *for which* $F(\theta, b)$ *vanishes*, and again functions for which *no value of* θ *satisfying the equation* $F(\theta, b) = 0$ *lies inside the circle defined by the inequality*

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

The function e^{ζ}, where

$$\zeta = 2\pi i \frac{z - a}{b - a},$$

* For a clear statement of this theorem and references see *Encyclopädie der mathematischen Wissenschaften*, II B 1, p. 103.

is an example of the former sort, while an illustration under the latter case is furnished by the polynomial

$$\left(\frac{z-a}{b-a}-\gamma\right)^m,$$

where m is an integer greater than 3, and γ is the solution

$$\gamma=\tfrac{1}{2}\left(1+i\cot\frac{\pi}{m}\right)$$

of the equation

$$(\gamma-1)^m=\gamma^m.$$

For this function, $\theta=\gamma$ is the only solution of the equation $F(\theta, b)=0$, and we have

$$|\gamma-\tfrac{1}{2}|=\tfrac{1}{2}\cot\frac{\pi}{m}\geqq\tfrac{1}{2}.$$

A closely related question is involved in the discussion of the equation

$$(5)\qquad F(\theta, z_1, z_2)\equiv f(z_1)-f(z_2)-(z_1-z_2)\,f'[z_2+\theta\,(z_1-z_2)]=0,$$

where both z_1 and z_2 are variable. If $f''(a)\neq 0$, we can make use of an auxiliary function defined by the equations

$$(6)\qquad \Phi(\theta, z_1, z_2)=\frac{F(\theta, z_1, z_2)}{(z_1-z_2)^2},\quad z_1\neq z_2,$$

$$\Phi(\theta, z_1, z_1)=(\tfrac{1}{2}-\theta)\,f''(z_1).$$

This function vanishes at the point $(\frac{1}{2}, a, a)$ and is analytic in the neighborhood of that point, while

$$\frac{\partial}{\partial\theta}\Phi(\tfrac{1}{2}, a, a)\neq 0.$$

We can therefore state the following theorem:

THEOREM II. *Let a be an interior point of a region B throughout which $f(z)$ is analytic, while $f''(a)\neq 0$. Then there exists a neighborhood A of a such that for all values of z_1 and z_2 in A the equation*

$$(5)\qquad f(z_1)-f(z_2)-(z_1-z_2)\,f'[z_2+\theta\,(z_1-z_2)]=0$$

has a solution θ satisfying the inequality

$$|\theta-\tfrac{1}{2}|<\tfrac{1}{2}.$$

If $f''(a) = 0$, the auxiliary function Φ defined by (6) cannot be used, and in fact *the above theorem may be false*. For example, if $f(z) = z^m$, where m is an integer greater than 3, it is possible to find points z_1, z_2 in every neighborhood of the origin such that all solutions of (5) satisfy the relation

$$|\theta - \tfrac{1}{2}| \geqq \tfrac{1}{2}.$$

We have this result when $z_1 = e^{\frac{2i\pi}{m}} z_2$, since the only solution of (5) is then

$$\theta = \frac{1}{1 - e^{\frac{2i\pi}{m}}} = \gamma.$$

3. Formulas Involving Integrals. As a preliminary to the development of theorems more general than the preceding, we shall now consider the equation

$$(7) \quad F_1(\theta, z) \equiv \int_a^z f(w, z)\ \phi(w, z)\, dw - f(a, z) \int_a^{a+\theta(z-a)} \phi(w, z)\, dw = 0,$$

which is analogous to the formula of Bonnet's theorem.* The functions $f(w, z)$, $\phi(w, z)$ will be taken as analytic functions of both w and z for all values of those variables satisfying inequalities

$$|z - a| < h, \qquad |w - a| < k,$$

while $f(a, z)$ will be supposed not identically zero in the region defined by the former of the above inequalities.

For the functions $f(w, z)$, $\phi(w, z)$ we have developments

$$f(w, z) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \alpha_{\mu\nu} (w - a)^\mu (z - a)^\nu,$$

$$\phi(w, z) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \beta_{\mu\nu} (w - a)^\mu (z - a)^\nu.$$

By substituting these developments in (7), we obtain for $F_1(\theta, z)$ the expansion

$$(8) \qquad F_1(\theta, z) = \sum_{\lambda=0}^{\infty} P_{\lambda+1}(\theta)(z - a)^{\lambda+1},$$

* *Journal de Mathématiques*, vol. 14 (1849), p. 249.

where

$$(9)\quad P_{\lambda+1}(\theta) \equiv \sum_{\mu=0}^{\lambda} \frac{1}{\mu+1} \sum_{\sigma=0}^{\lambda-\mu} \Big[\sum_{\rho=0}^{\mu} a_{\rho\sigma}\beta_{\mu-\rho,\,\lambda-\mu-\sigma} - \theta^{\mu+1} a_{0\sigma}\beta_{\mu,\,\lambda-\mu-\sigma} \Big].$$

For any value $\theta = \theta_1$, and for any positive real number K, there exists a positive real number H such that the series (8) converges throughout the region defined by the inequalities

$$|\theta - \theta_1| < K, \qquad |z - a| < H.$$

The following theorem can now be established:

THEOREM III. *Let $f(w, z)$ and $\phi(w, z)$ be analytic functions of both w and z throughout a region defined by inequalities*

$$|z - a| < h, \qquad |w - a| < k,$$

and let λ_1 be the least value of λ for which the polynomial $P_{\lambda+1}(\theta)$ does not vanish identically. Then if T is any region in the θ-plane of which a root θ_1 of the equation $P_{\lambda_1+1}(\theta) = 0$ is an interior point, there exists a neighborhood A of the point $z = a$ such that the equation (7) is satisfied by a value of θ in T for every value of z in A.

The auxiliary function to be used here is obvious. If θ_1 is not a multiple root of $P_{\lambda_1+1}(\theta) = 0$, so that

$$\frac{\partial}{\partial\theta} P_{\lambda_1+1}(\theta_1) \neq 0,$$

we use the same existence theorem for implicit functions as in the proof of theorem I. In the case of a multiple root we avail ourselves of the more general existence theorem which may be found, for example, in Picard's *Traité d'analyse*, vol. 2 (2d. edit.), p. 261.

From theorem III we can deduce formulas which correspond to familiar theorems of mean value. In the one which follows, an analogue of Bonnet's theorem, it is interesting to note that as z approaches a, θ approaches a point *on* the unit circle instead of a point in its interior as in theorems I, II, and V.

THEOREM IV. *Let $f(z)$ and $\phi(z)$ be analytic throughout a region B of which a is an interior point. Then there exists a neighborhood A of a such that for any value of z in A the equation*

$$(10)\qquad \int_a^z f(w)\phi(w)dw - f(a)\int^{a+\theta(z-a)} \phi(w)dw = 0$$

has a solution θ satisfying the inequality

$$|\theta - 1| < 1.$$

Equation (10) is a special case of (7) where f and ϕ are functions of but one variable, so that

$$a_{\mu\nu} = \beta_{\mu\nu} = 0, \quad \nu \neq 0.$$

By hypothesis

$$f(a) = a_{00} \neq 0,$$

hence, if $\beta_{\lambda_1 0}$ is the first non-vanishing coefficient $\beta_{\lambda 0}$, we have

$$P_{\lambda_1+1}(\theta) = \frac{a_{00}\beta_{\lambda_1 0}}{\lambda_1 + 1}(1 - \theta^{\lambda_1+1}).$$

Since $\theta = 1$ is always a root of $P_{\lambda_1+1}(\theta) = 0$, theorem IV follows from theorem III, the special region T assigned in IV having been chosen in order to present a formula as closely analagous as possible to the ordinary Bonnet formula.

The two formulas of the following theorem are usually referred to as the first and second theorems of mean value for integrals:

THEOREM V. *Let $f(z)$ and $\phi(z)$ be analytic throughout a region B of which a is an interior point. Then for each of the equations*

$$(11) \qquad \int_a^z f(w)\phi(w)dw - f\Big(a + \theta(z-a)\Big)\int_a^z \phi(w)dw = 0,$$

$$(12) \qquad \int_a^z f(w)\phi(w)dw - f(a)\int_a^{a+\theta(z-a)} \phi(w)dw - f(z)\int_{a+\theta(z-a)}^z \phi(w)dw = 0,$$

there exists a neighborhood A of a such that for any value of z in A the corresponding equation has a solution θ satisfying the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

To deduce this result for formula (11) we make the following substitutions in equation (7):

$$f(w, z) = \int_w^z \phi_1(w)dw,$$

$$\phi(w, z) = f_1'(w).$$

Integration by parts now gives to (7) the form

$$F_1(\theta, z) \equiv \Big[f_1(w)\int_w^z \phi_1(w)dw\Big]_{w=a}^{w=z} + \int_a^z f_1(w)\phi_1(w)dw$$
$$- \Big[f_1[a+\theta(z-a)] - f_1(a)\Big]\int_a^z \phi_1(w)dw$$
$$\equiv \int_a^z f_1(w)\phi_1(w)\,dw - f_1[a+\theta(z-a)]\int_a^z \phi_1(w)dw = 0.$$

We have only to drop subscripts to obtain equation (11), and the part of our theorem which relates to this equation is demonstrated if we can show that a root of $P_{\lambda_1+1}(\theta) = 0$ lies in the region defined by the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

The functions $f(w)$ and $\phi(w)$ will have developments

$$f(w) = \sum_{\nu=0}^{\infty} a_\nu (w-a)^\nu,$$
$$\phi(w) = \sum_{\nu=0}^{\infty} \beta_\nu (w-a)^\nu. \tag{13}$$

If $f(w)$ is a constant, or if $\phi(w) \equiv 0$, equation (11) is obviously an identity. In any other case let a_m and β_k be the first non-vanishing numbers of the respective sequences

$$a_1, a_2, \cdots,$$
$$\beta_0, \beta_1, \cdots.$$

We then have

$$\lambda_1 = m + k,$$

$$P_{\lambda_1+1}(\theta) = a_m\beta_k\Big(\frac{1}{m+k+1} - \frac{\theta^m}{k+1}\Big),$$

and this last equation always has a real root θ_1 satisfying the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

Equation (12) may be derived from (7) by the substitution

$$f(w, z) = f(w) - f(z),$$
$$\phi(w, z) = \phi(w).$$

As in the discussion of equation (11), we note that if $f(w)$ is a constant, or if $\phi(w) \equiv 0$, (12) is an identity. In any other case a_m and β_k have the same meaning as before, when $f(w)$ and $\phi(w)$ have the developments (13). We find

$$\lambda_1 = m + k,$$

$$P_{\lambda_1+1}(\theta) = -\frac{a_m\beta_k}{k+1}\left[\frac{m}{m+k+1} - \theta^{k+1}\right],$$

so that in this case also $P_{\lambda_1+1}(\theta) = 0$ has a root between 0 and 1 on the axis of reals.

Cauchy's formula*

$$\frac{f(z) - f(a)}{\phi(z) - \phi(a)} = \frac{f'[a + \theta(z-a)]}{\phi'[a + \theta(z-a)]} \qquad |\theta - \tfrac{1}{2}| < \tfrac{1}{2},$$

and the various forms of Taylor's series with remainder are special cases of the first formula of theorem V. In particular, theorem I is a corollary of this part of V.

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* *Calcul differential*, p. 37.

NOTE ON REGULAR POLYGONS*

By Charlotte Angas Scott

It is a well-known fact that the circle described with its centre at any point P of a rectangular hyperbola and passing through the other extremity of the diameter through P cuts the hyperbola again at the vertices of an equilateral triangle. The object of the present note is to show that the circles which cut a rectangular hyperbola at four of the vertices of a regular pentagon, heptagon, or nonagon can be described with equal facility.

It is necessary, in the first place, to find a rectangular hyperbola that shall cut a given circle, $x^2 + y^2 = 1$, at four vertices of one of these polygons.

1. The Heptagon. If a side of the heptagon subtends at the centre an angle θ, then $7\theta = 2\pi$, hence $\sin 3\theta = -\sin 4\theta$; that is,

$$3 \sin \theta - 4 \sin^3 \theta = -4 \sin \theta \cos \theta \, (2 \cos^2 \theta - 1),$$

from which,

$$\sin \theta \{8 \cos^3 \theta + 4 \cos^2 \theta - 4 \cos \theta - 1\} = 0.$$

If one vertex is at $(1, 0)$ (see figure 1) the roots of this equation give the values of θ for the vertices. If x is written for $\cos \theta$, the solution $\sin \theta = 0$, which refers to the vertex $(1, 0)$ and gives therefore $\theta = 0$, becomes $x - 1 = 0$; and the cubic factor gives

$$8x^3 + 4x^2 - 4x - 1 = 0.$$

Hence the roots of

$$(x - 1)(8x^3 + 4x^2 - 4x - 1) = 0,$$

that is, of

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0,$$

are the abscissæ of the vertices numbered 7, 1 and 6, 2 and 5, 3 and 4, in figure 1.

It is desired to find a rectangular hyperbola that shall meet the circle at the point 7, and at one point of every pair $(1, 6)$, $(2, 5)$, $(3, 4)$. There are clearly 2^3, that is, 8 such rectangular hyperbolas; any one will serve the purpose.

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* Presented to the American Mathematical Society, October 27, 1906.

The general equation of a rectangular hyperbola is

$$ax^2 + 2hxy - ay^2 + 2gx + 2fy + c = 0.$$

To find where this meets the circle

$$x^2 + y^2 - 1 = 0,$$

eliminate y. The two equations give, by elimination of the term y^2,

$$2ax^2 + 2hxy + 2gx + 2fy + c - a = 0,$$

that is,

$$2ax^2 + 2gx + c - a = -y\,(2hx + 2f);$$

hence, since

$$-y^2 = x^2 - 1,$$

$$(x^2 - 1)(2hx + 2f)^2 + (2ax^2 + 2gx + c - a)^2 = 0.$$

The equation for the abscissæ of the common points is therefore

$$(4h^2 + 4a^2)\,x^4 + (8hf + 8ag)\,x^3 + \{4f^2 - 4h^2 + 4g^2 + 4a(c-a)\}x^2 + \{-8hf + 4g(c-a)\}x - 4f^2 + (c-a)^2 = 0,$$

and this is to be the same as

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0.$$

Hence, equating coefficients,

$$4h^2 + 4a^2 = 8\lambda, \tag{1}$$

$$8hf + 8ag = -4\lambda, \tag{2}$$

$$4f^2 - 4h^2 + 4g^2 + 4a(c-a) = -8\lambda, \tag{3}$$

$$-8hf + 4g(c-a) = 3\lambda, \tag{4}$$

$$-4f^2 + (c-a)^2 = \lambda. \tag{5}$$

The sum of these gives

$$4g^2 + 4g(c+a) + (c+a)^2 = 0,$$

that is,

$$2g + c + a = 0.$$

The sum of (2) and (4) gives

$$4g(c + a) = -\lambda.$$

Hence $$-2g = a + c, \qquad \lambda = 2(a + c)^2.$$

With these values for g and λ, (1), (5), (2) become

$$h^2 = 4(a + c)^2 - a^2,$$

$$4f^2 = (c - a)^2 - 2(a + c)^2 = -(a + c)^2 - 4ac,$$

$$2hf = a(a + c) - 2(a + c)^2.$$

Hence, in virtue of the identity $h^2 \times 4f^2 = (2hf)^2$, we have

$$\{4(a + c)^2 - a^2\}\{-(a + c)^2 - 4ac\} = (a + c)^2\{a - 2(a + c)\}^2.$$

Since $2(a + c) - a$ is a factor of each side of this equation, one solution is $2(a + c) - a = 0$, that is,

$$a = -2c.$$

This leads to

$$2g = -(a + c) = c,$$

$$\lambda = 2(a + c)^2 = 2c^2,$$

$$h^2 = 4(a + c)^2 - a^2 = 4c^2 - 4c^2 = 0,$$

$$4f^2 = -(a + c)^2 - 4ac = -c^2 + 8c^2 = 7c^2.$$

Since c is a factor in all the coefficients, it is not zero; hence we may give it any convenient value, for example, -1. Then $a = 2$, $h = 0$, $2g = -1$, $2f = \pm\sqrt{7}$; choose for $2f$ the value $+\sqrt{7}$.

The four possible values for $a : c$, and the two for f, account for the eight solutions of the problem. The other three values for $a : c$, given by the cubic

$$\{2(a + c) + a\}\{-(a + c)^2 - 4ac\} = (a + c)^2\{2(a + c) - a\},$$

are irrational. Since only one solution out of the eight is needed, it is unnecessary to attend to these.

The rectangular hyperbola is

$$2x^2 - 2y^2 - x + \sqrt{7}\, y - 1 = 0.$$

The center is at $(\frac{1}{4}, \frac{1}{4}\sqrt{7})$; when the origin is transferred to this point the equation becomes

$$x^2 - y^2 = \frac{1}{8},$$

hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in figure 1; it passes through the vertices 1, 2, 4, 7.

Since the foci of $x^2 - y^2 = \frac{1}{8}$ are $(\pm \frac{1}{2}, 0)$, with the original axes they are $(\frac{3}{4}, \frac{1}{4}\sqrt{7})$, $(-\frac{1}{4}, \frac{1}{4}\sqrt{7})$; of these the first lies on the circle. The corresponding directrices are $x = \pm \frac{1}{4}$, that is, referred to the original axes, $x = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $x = -\frac{1}{4} + \frac{1}{4} = 0$; the second of these passes through the centre of the circle. The distance between the foci is equal to the radius of the circle.

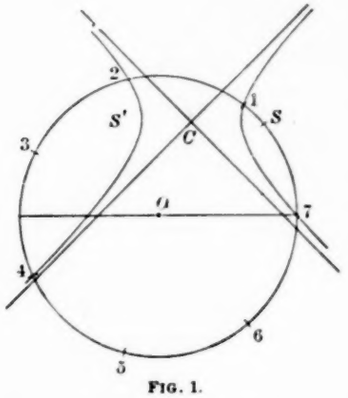

FIG. 1.

2. The Nonagon. In the case of the nonagon, the angle θ is determined by $3\theta = \frac{2}{3}\pi$; hence $\cos 3\theta = -\frac{1}{2}$, that is,

$$4\cos^3\theta - 3\cos\theta + \tfrac{1}{2} = 0,$$

or

$$8x^3 - 6x + 1 = 0.$$

The roots of this equation are the abscissæ of the vertices 1, 4, 7 (see figure 2), hence of the pairs of vertices (1, 8), (4, 5), (7, 2). To bring in 3 and 6, for which $x = -\frac{1}{2}$, we multiply by $2x + 1$; the equation becomes

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

By the same argument as in the case of the heptagon, the coefficients in

the equation of the desired rectangular hyperbola are determined by the five equations

$$\begin{aligned} 4h^2 + 4a^2 &= 16\lambda, && (1)\\ 8hf + 8ag &= 8\lambda, && (2)\\ 4f^2 - 4h^2 + 4g^2 + 4a(c-a) &= -12\lambda, && (3)\\ -8hf + 4g(c-a) &= -4\lambda, && (4)\\ -4f^2 + (c-a)^2 &= \lambda. && (5) \end{aligned}$$

These, combined exactly as before, give

$$4g^2 + 4g(c+a) + (c+a)^2 = 9\lambda,$$

$$4g(c+a) = 4\lambda,$$

hence $$4g^2 - 5g(c+a) + (c+a)^2 = 0,$$

that is, $$\{4g - (c+a)\}\{g - (c+a)\} = 0$$

from which $$4g = c + a \quad \text{or} \quad g = c + a.$$

Of these, $4g = c + a$ leads to the simplest results; we find $4\lambda = (c+a)^2$, and then equations (1), (5), (2) become

$$h^2 = (a+c)^2 - a^2 = c(2a+c),$$

$$16f^2 = 4(c-a)^2 - (a+c)^2 = (a-3c)(3a-c),$$

$$4hf = (a+c)^2 - a(a+c) = c(a+c).$$

Hence, by means of $h^2 \times 16f^2 = (4hf)^2$,

we have

$$c(2a+c)(a-3c)(3a-c) = c^2(a+c)^2.$$

One solution of this equation is $c = 0$; the other three are irrational. We take therefore $c = 0$; then $h = 0$, $4g = a$, $16f^2 = 3a^2$, from which $4f = \pm\sqrt{3}\,a$; take $4f = +\sqrt{3}\,a$.

(Since there are four pairs of vertices, the number of solutions to be expected is 2^4, that is 16. All are accounted for; two values for g, four for $a : c$, two for f, give $2 \times 4 \times 2 = 16$ solutions).

Since a is a factor in all the coefficients, it is not zero; hence we may assign any convenient value, in this case 2. The desired rectangular hyperbola is

$$2x^2 - 2y^2 + x + \sqrt{3}\,y = 0.$$

The centre is at $(-\frac{1}{4}, \frac{1}{4}\sqrt{3})$; when the origin is transferred to this point the equation becomes

$$-x^2 + y^2 = \frac{1}{8},$$

hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in figure 2; it passes through the vertices 2, 3, 5, 8, and through the centre of the circle.

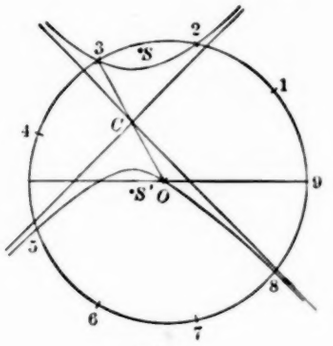

FIG. 2.

The radius of the circle to the vertex 3 is a diameter of the hyperbola; the distance between the foci is equal to the radius of the circle.

The rectangular hyperbola thus obtained for the nonagon is of the same size as that obtained for the heptagon, but differently placed.

3. The Pentagon. In the case of the pentagon, $5\theta = 2\pi$, hence $\sin 2\theta = -\sin 3\theta$,

from which

$$2 \sin\theta \cos\theta + 3 \sin\theta - 4 \sin^3\theta = 0;$$

that is,

$$\sin\theta = 0 \quad \text{or} \quad 2\cos\theta + 3 - 4\sin^2\theta = 0.$$

Hence

$$4\cos^2\theta + 2\cos\theta - 1 = 0,$$

that is,

$$4x^2 + 2x - 1 = 0.$$

The roots of this quadratic are the abscissæ of the pairs of vertices (1, 4), (2, 3) see figure 3. A rectangular hyperbola, with the axis of x as one axis of symmetry, can be passed through the four points. The equation of such a hyperbola is

$$ax^2 - ay^2 + 2gx + c = 0;$$

the abscissæ of the intersections with the circle are the roots of

$$ax^2 + a(x^2 - 1) + 2gx + c = 0,$$

that is, of

$$2ax^2 + 2gx - (a - c) = 0.$$

Hence $2a = 4$, $2g = 2$, $a - c = 1$, and the hyperbola is therefore

$$2x^2 - 2y^2 + 2x + 1 = 0.$$

The centre is at $(-\frac{1}{2}, 0)$; when this point is taken as origin the equation becomes

$$x^2 - y^2 + \tfrac{1}{4} = 0,$$

hence the semi-axis major is $\frac{1}{2}$. The curve lies as shown in figure 3. The radius of the circle to the point $(-1, 0)$ is the minor axis of the hyperbola.

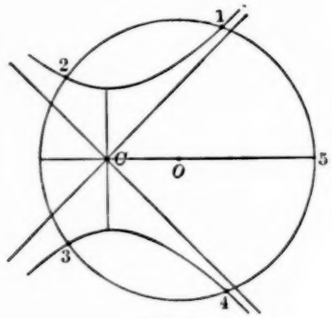

FIG. 3.

4. Conclusion. If we change the origin in all three cases to the centre of the hyperbola, and change the scale so that the hyperbola has the semi-axis major = 1; and moreover, in the case of the heptagon, interchange the axes of x and y; we arrive at the conclusion that the hyperbola $x^2 - y^2 + 1 = 0$ is cut at four vertices of a regular pentagon by the circle $(x - 1)^2 + y^2 = 4$; at four vertices of a regular heptagon by the circle

$$\left(x - \sqrt{\frac{7}{2}}\right)^2 + \left(y + \frac{1}{\sqrt{2}}\right)^2 = 8;$$

at four vertices of a regular nonagon by the circle

$$\left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(y + \sqrt{\frac{3}{2}}\right)^2 = 8.$$

The relation of the different circles to the hyperbola is, however, more neatly expressed by the geometrical statements of the results:

(1) A rectangular hyperbola is cut at four vertices of a regular pentagon by a circle on the minor axis as a radius (figure 4).

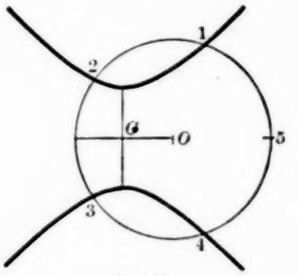

FIG. 4.

(2) A rectangular hyperbola is cut at four vertices of a regular heptagon by a circle of radius equal to the distance between the foci, with its centre on one directrix and passing through the other focus (figure 5).

(3) A rectangular hyperbola is cut at four vertices of a regular nonagon by a circle which has for radius a diameter of the hyperbola of length equal to the distance between the foci (figure 6).

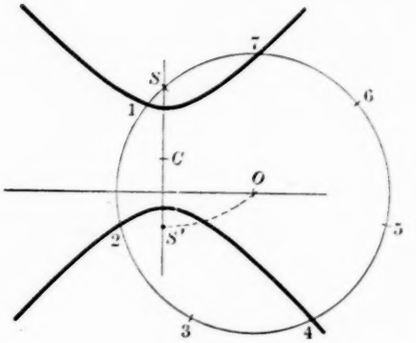

FIG. 5.

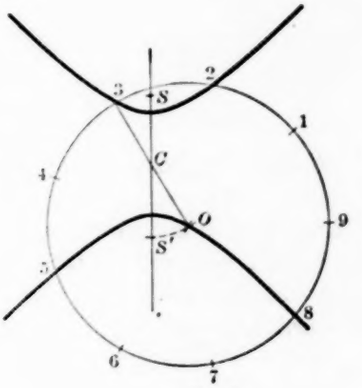

FIG. 6.

BRYN MAWR COLLEGE,
BRYN MAWR, PENNSYLVANIA,
OCTOBER, 1906.

THE REVOLUTION OF A DARK PARTICLE ABOUT A LUMINOUS CENTER

By Edwin Bidwell Wilson

1. Introduction. The simple problem of two bodies moving under their mutual gravitational attraction has doubtless long been considered as completely solved. Yet ever since Maxwell's formulation of his theory of electromagnetic action as a motion or stress of some kind in the ether, there has been a remote possibility that the time might come for revising the dynamical equations of motion so as to take into account the action of the ether on the moving masses. In fact, recent investigators of the electronic theories of matter have paid a great deal of attention to the interaction of matter and ether. To adapt their work to the problem of the revolution of bodies about a fixed luminous center, which is approximately the case that obtains in the solar system, is by no means easy. For in the first place there are different theories of electrons and in the second place these theories lead one to try to replace the older and more familiar dynamical equations by newer and as yet not fully settled electrodynamical equations. Thus Abraham* discusses in considerable detail the actions of the ether on a moving body; but his equations are those of the electromagnetic field and depend to some extent on his particular theory.

Recently, however, Poynting, with that directness of insight and sharp mechanical instinct which characterize the English investigators of physical science, has taken up the problem of the effect of radiation on moving masses and has treated it from the point of view of the transmission of momentum by the ether. His results, although perhaps essentially electromagnetic in that they are derived from the consideration of waves in the ether, are nevertheless well adapted to direct mechanical formulation.† The effects which he describes

* *Theorie der Elektrizität*, vol. 2.

† We do not wish to be interpreted as implying either that the pressure of radiation is *necessarily* connected with any electromagnetic theory or that Poynting was the first to consider the matter from other points of view. On the contrary the question is rather one of mechanics and thermo dynamics and had previously been treated as such by several authors. Historical references may be found in an article on "The pressure due to radiation" by Nichols and Hull, *Astrophysical Journal*, vol. 17, June, 1903 or *The Smithsonian Report for* 1903, pp. 115–138. Poynting merely reverts to this simpler treatment and applies it anew.

are three:* the direct pressure of light, the Doppler reception effect, and the Doppler emission effect. We will state what is meant by each.

If a body which is under the rays of the sun be considered as dark† or essentially black, to use the technical term, the radiation from the sun is stopped and absorbed by the body. The result is an outward pressure due to the abstraction of momentum from the ether. As the electromagnetic radiation varies in its intensity according to the same law as the gravitational attraction, it produces a pressure which varies inversely as the square of the distance and hence introduces no change in the dynamical equations except an apparently diminished value for the gravitational action. It might seem at first sight that in estimating the pressure due to radiation the distance should be measured from the radiating surface: but a moment's consideration shows that a spherical body like the sun could be replaced by a concentric body radiating with a properly altered intensity, and hence that the distance should be measured from the center.

The two Doppler effects are much smaller than the direct pressure of light. To consider the reception effect suppose that the dark body were moving toward the sun or any other luminous center with a velocity v. Then a greater amount of momentum would be absorbed per unit time than if the body were at rest. In particular the amount would be greater in the ratio v/V, where V denotes the velocity of light. In case the body falls to the sun from an infinite distance this ratio is about 2/1000, which may be considered as its maximum value in the solar system; generally it is much smaller. If the body retreats, the pressure is lessened in the same ratio. Finally there is the emission effect which arises from the fact that the dark body radiates the energy it receives. This force, which is "frictional" in that it always opposes the

* See his address to the Royal Society, *Nature*, vol. 75, pp. 90–93, November 22, 1906. Earlier than this, Poynting had printed an extensive article on "Radiation in the solar system: its effect on temperature and its pressure on small bodies," *Transactions of the Royal Society*, ser. A, vol. 202 (1903), pp. 525–552. In this article there occur some minor errors which the author corrects in a note at the end. Moreover the two Doppler effects are by no means clearly distinguished one from another. Finally in this earlier treatment, *loc. cit.*, p. 549, the author comes upon a differential equation that he states is probably not integrable This equation corresponds to our equation (5). The integration which we give in terms of Bessel functions not only furnishes a desirable complete determination of the problem, but enables us to see certain other general characteristics of the motion than are obtainable from a less exact integration.

† This is approximately true for the small particles in the solar system. As far as we know them, their albedo is very low.

motion, is stated by Poynting to be one-third of v/V of the direct pressure of light.* This does not make the emission effect merely one-third as great as the reception effect: for the velocity which enters into the determination of the reception effect is the component velocity of the motion resolved along the line joining the particle and the sun, whereas in the emission effect it is the total velocity in path that enters into the formula.

2. The differential equations of motion and their integrals. These statements of Poynting's may be incorporated in the dynamical equations in a simple manner if polar coordinates with the sun as center be assumed for reference. Let $-C/r^2$ and K/r^2 respectively be the accelerations due to the gravitational attraction and the direct pressure of radiation. There will be the accelerations

$$-\frac{K}{r^2}\frac{1}{V}\frac{dr}{dt} \qquad \text{and} \qquad -\frac{1}{3}\frac{K}{r^2}\frac{1}{V}\frac{dr}{dt}$$

along the radius vector due respectively to the reception and emission effects. Further there will be the single acceleration

$$-\frac{1}{3}\frac{K}{r^2}\frac{1}{V}\frac{rd\theta}{dt}$$

perpendicular to the radius. The equations of motion then become †

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{C-K}{r^2} - \frac{4}{3}\frac{K}{Vr^2}\frac{dr}{dt}, \tag{1}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = -\frac{1}{3}\frac{K}{Vr^2}\frac{rd\theta}{dt}. \tag{2}$$

The simplicity of these equations is largely due to the fortunate linear

* As a matter of fact Poynting states, as is well known, that higher powers of v/V enter into the expression; but neglecting them does not introduce, on the assumptions made above, an error of more than one-fifth of one per cent in a quantity already small; and generally the error is much less than this. To introduce the exact expression for the emission effect (which is evaluated on assumptions that are not perfectly well established) would unnecessarily complicate the equations of motion.

† Provided we disregard, as we may, any changes in the mass of the particle which might arise from its motion through the electromagnetic field. On the electronic theories there is such an effect, but it remains very small until the velocity of the moving body is within a few percent of the velocity of light.

dependence of the Doppler effects on the velocity of the particle. Equation (2) is immediately integrable into

$$r^2 \frac{d\theta}{dt} = h - \frac{K}{3V}\theta. \tag{3}$$

To find the orbit of the particle we may eliminate the time between this equation and (1). Meanwhile in order to reduce the resulting equation to a simpler form, it will be well to introduce the change of variables

$$u = \frac{1}{r}, \quad \frac{3Vh}{K} - \theta = \phi. \tag{4}$$

The differential equation of the orbit then takes the particularly simple form

$$\frac{d^2u}{d\phi^2} - \frac{3}{\phi}\frac{du}{d\phi} + u = \frac{9V^2}{K^2}\frac{C-K}{\phi^2} = \frac{L}{\phi^2}. \tag{5}$$

Two things may be noted at this point. First, if K be allowed to approach zero, ϕ becomes infinite in such a manner that $K\phi$ remains finite and $d\phi = -d\theta$. Hence (5) approaches, as it should, the form

$$\frac{d^2u}{d\theta^2} + u = \text{const.},$$

ordinarily obtained in the theory of gravitation for the revolution of a planet. Secondly it is noteworthy that (5) is, except for the presence of the second member, an equation of the Bessel type.

To bring the differential equation of the orbit into shape for easier comparison with the treatises on Bessel functions the change of variable $u = z\phi^3$ will be made. Then (5) takes the final form:

$$\frac{d^2z}{d\phi^2} + \frac{1}{\phi}\frac{dz}{d\phi} + \left(1 - \frac{2^2}{\phi^2}\right)z = \frac{L}{\phi^4}. \tag{6}$$

If the second member were zero, the general solution would be

$$z = AJ_2(\phi) + BY_2(\phi)$$

in terms of Bessel functions J and Neumann functions Y.* It remains to find a particular solution of the complete equation to add to this. These particular

*The notations throughout this article will conform to those of the *Treatise on Bessel Functions* by Gray and Mathews (1895).

solutions, rarely given in the books, may readily be obtained by the method of variation of constants. In this case such a solution is

$$L\left[Y_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Y_2(\phi)}{\phi^3}\,d\phi\right].$$

Hence the equation of the orbit may be written as

$$(7)\quad u=\phi^2\left\{AJ_2(\phi)+BY_2(\phi)+L\left[Y_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Y_2(\phi)}{\phi^3}\,d\phi\right]\right\},$$

where the constants A and B may be chosen so that the lower limit of integration may take any desired value.

As the Doppler effects are very small, the motion of the particle will be nearly elliptic, parabolic, or hyperbolic. To bring out more clearly the departures from these forms we may expand the solution (7) by means of the asymptotic expressions for the Bessel and Neumann functions. These series will semi-converge with extreme rapidity because of the large value of the argument ϕ (we shall have no occasion to consider cases in which ϕ is materially less than 10^6 as may be seen by reference to p. 148). For such cases as these the function Y is inconvenient: it contains an additive multiple of J. Let us write *

$$Y_n(\phi)=(\log 2-\gamma)\,J_n(\phi)+Z_n(\phi).$$

Then the function Z satisfies the fundamental relations which J and Y satisfy and which are the reduction formulas† especially useful in the manipulation of these functions. We have then the asymptotic formulas

$$\left(\frac{\pi\phi}{2}\right)^{1/2}J_n(\phi)=P\cos\left\{\frac{(2n+1)\pi}{4}-\phi\right\}+Q\sin\left\{\frac{(2n+1)\pi}{4}-\phi\right\},$$

$$\left(\frac{2\phi}{\pi}\right)^{1/2}Z_n(\phi)=Q\cos\left\{\frac{(2n+1)\pi}{4}-\phi\right\}-P\sin\left\{\frac{(2n+1)\pi}{4}-\phi\right\},$$

where

$$P=1-\frac{(4n^2-1)(4n^2-9)}{2!(8\phi)^2}+\text{higher even powers of }\frac{1}{\phi},$$

$$Q=\frac{4n^2-1}{8\phi}-\frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8\phi)^3}+\text{higher odd powers of }\frac{1}{\phi}.$$

* See Gray and Mathews, p. 40, formula 95.

† These are: $Z'_n=(n/\phi)Z_n-Z_{n+1}=Z_{n-1}-(n/\phi)Z_n$, $Z_{n+1}-(2n/\phi)Z_n+Z_{n-1}=0$, $J_{n+1}Z_n-J_nZ_{n+1}=0$, and $(\phi^nZ_n)'=\phi^nZ_{n-1}$. See Gray and Mathews, p. 14 and p. 16.

Or if we set $(2n+1)\,\pi/4 - \phi = \psi_n$ we may write

$$\left(\frac{\pi\phi}{2}\right)^{1/2} J_n(\phi) = \cos\psi_n + \frac{n^2-\frac{1}{4}}{2\phi}\sin\psi_n - \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2!(2\phi)^2}\cos\psi_n + \cdots,$$

(8)

$$\left(\frac{2\phi}{\pi}\right)^{1/2} Z_n(\phi) = -\sin\psi_n + \frac{n^2-\frac{1}{4}}{2\phi}\cos\psi_n + \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{2!(2\phi)^2}\sin\psi_n + \cdots,$$

with the relations $\cos\psi_{n-1} = \sin\psi_n$, $\sin\psi_{n-1} = -\cos\psi_n$. The terms omitted from (8) are of the order 10^{-18} when ϕ is of the order 10^6.

In terms of the functions J and Z the equation of the orbit becomes

$$(9)\quad u = \phi^2\left\{CJ_2(\phi) + DZ_2(\phi) + L\left[Z_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Z_2(\phi)}{\phi^3}\,d\phi\right]\right\},$$

where C and D are constants of integration slightly different from A and B. This form of the equation is not yet in the shape most convenient for computation. For the integrals in the bracket change their value by a hundred percent when ϕ runs through an interval of 2π. It may be noted, however, that the maximum value of the integrals, when extended over a small interval in ϕ, is very small owing to the fact that the denominator remains of the order 10^{18} and the numerator of order 10^{-3}. An integration by parts will enable us to accentuate this. We have

$$\int\frac{J_2(\phi)}{\phi^3}\,d\phi = \int\frac{\phi^3 J_2(\phi)}{\phi^6}\,d\phi = \frac{\phi^3 J_3(\phi)}{\phi^6} + 6\int\frac{\phi^3 J_3(\phi)}{\phi^7}\,d\phi = \frac{J_3(\phi)}{\phi^3} + 6\int\frac{\phi^4 J_3(\phi)}{\phi^8}\,d\phi.$$

Hence

$$\int\frac{J_2(\phi)}{\phi^2}\,d\phi = \frac{J_3(\phi)}{\phi^3} + 6\,\frac{J_4(\phi)}{\phi^4} + 6\cdot 8\,\frac{J_5(\phi)}{\phi^5} + \cdots,$$

with a similar result for the integral of $Z_2(\phi)/\phi^3$. Combining, we have for the bracket

$$Z_2(\phi)\int\frac{J_2(\phi)}{\phi^3}\,d\phi - J_2(\phi)\int\frac{Z_2(\phi)}{\phi^3}\,d\phi = \frac{1}{\phi^4} - \frac{12}{\phi^6} + \text{higher even powers of } \frac{1}{\phi}.$$

There appears to be no necessity of keeping terms of the order 10^{-48}. Hence

$$(10)\qquad u = \phi^2\left\{CJ_2(\phi) + DZ_2(\phi) + L\left(\frac{1}{\phi^4} - \frac{12}{\phi^6}\right)\right\}$$

may be taken as the approximate equation of the orbit for future computations. For the derivative we have, from (10), the expression

$$u' = \phi^2\left\{CJ_1(\phi) + DZ_1(\phi) - L\frac{2}{\phi^5}\right\}, \tag{11}$$

where the small quantity $48L/\phi^7$ has been neglected within the brace.

If the initial values u_0, u'_0, and ϕ_0 of u, u', and ϕ be substituted in (10) and (11), the constants C and D may be obtained.

$$C = \frac{1}{\phi_0}\left\{\left[u_0 - L\left(\frac{1}{\phi_0^2} - \frac{12}{\phi_0^4}\right)\right]Z_1(\phi_0) - \left[u'_0 + \frac{2L}{\phi_0^3}\right]Z_2(\phi_0)\right\}$$
$$= \frac{1}{\phi_0}\Big(RZ_1(\phi_0) - SZ_2(\phi_0)\Big),$$

$$D = \frac{1}{\phi_0}\left\{-\left[u_0 - L\left(\frac{1}{\phi_0^2} - \frac{12}{\phi_0^4}\right)\right]J_1(\phi_0) + \left[u'_0 + \frac{2L}{\phi_0^3}\right]J_2(\phi_0)\right\}$$
$$= \frac{1}{\phi_0}\Big(-RJ_1(\phi_0) + SJ_2(\phi_0)\Big).$$

In equations (8) let $\psi_n = \psi_2 = \psi$. Then we have, if $(2n+1)\pi/4 - \phi_0 = \psi_0$,

$$\left(\frac{\pi\phi_0}{2}\right)^{1/2}J_1(\phi_0) = \sin\psi_0 - \frac{\frac{3}{4}}{2\phi_0}\cos\psi_0 - \frac{\frac{3}{4}\cdot-\frac{5}{4}}{2!(2\phi_0)^2}\sin\psi_0 + \cdots,$$

$$\left(\frac{2\phi_0}{\pi}\right)^{1/2}Z_1(\phi_0) = \cos\psi_0 + \frac{\frac{3}{4}}{2\phi_0}\sin\psi_0 - \frac{\frac{3}{4}\cdot-\frac{5}{4}}{2!(2\phi_0)^2}\cos\psi_0 + \cdots.$$

Hence

$$Z_1(\phi_0)J_2(\phi) = \left(\frac{1}{\phi_0\phi}\right)^{1/2}\left[\cos\psi_0\cos\psi + \cos\psi_0\sin\psi\,\frac{\frac{15}{4}}{2\phi} + \sin\psi_0\cos\psi\,\frac{\frac{3}{4}}{2\phi_0}\right.$$
$$\left. + \sin\psi_0\sin\psi\,\frac{\frac{3}{4}\cdot\frac{15}{4}}{4\phi_0\phi} - \cos\psi_0\cos\psi\,\frac{\frac{15}{4}\cdot\frac{7}{4}}{2!(2\phi)^2} - \cos\psi_0\cos\psi\,\frac{\frac{3}{4}\cdot-\frac{5}{4}}{2!(2\phi_0)^2}\right],$$

and

$$J_1(\phi_0)Z_2(\phi) = \left(\frac{1}{\phi_0\phi}\right)^{1/2}\left[-\sin\psi_0\sin\psi + \sin\psi_0\cos\psi\,\frac{\frac{15}{4}}{2\phi} + \cos\psi_0\sin\psi\,\frac{\frac{3}{4}}{2\phi_0}\right.$$
$$\left. - \cos\psi_0\cos\psi\,\frac{\frac{3}{4}\cdot\frac{15}{4}}{4\phi_0\phi} + \sin\psi_0\sin\psi\,\frac{\frac{15}{4}\cdot\frac{7}{4}}{2!(2\phi)^2} + \sin\psi_0\sin\psi\,\frac{\frac{3}{4}\cdot-\frac{5}{4}}{2!(2\phi_0)^2}\right].$$

Hence

$$RZ_1(\phi_0)J_2(\phi) - RJ_1(\phi_0)Z_2(\phi) = R\left(\frac{1}{\phi_0\phi}\right)^{1/2}\Big[\cos(\psi_0-\psi) - \sin(\psi_0-\psi)\left(\frac{15}{8\phi}-\frac{3}{8\phi_0}\right) + \cos(\psi_0-\psi)\left(\frac{45}{64\phi_0\phi}-\frac{105}{128\phi^2}+\frac{15}{128\phi_0^2}\right)\Big].$$

In a similar manner

$$-SZ_2(\phi_0)J_2(\phi) + SJ_2(\phi_0)Z_2(\phi) = S\left(\frac{1}{\phi_0\phi}\right)^{1/2}\Big[\sin(\psi_0-\psi) + \cos(\psi_0-\psi)\left(\frac{15}{8\phi}-\frac{15}{8\phi_0}\right) + \sin(\psi_0-\psi)\left(\frac{225}{64\phi_0\phi}-\frac{105}{128\phi^2}-\frac{105}{128\phi_0^2}\right)\Big].$$

Now let $\psi_0 - \psi = \xi$. Then $\phi - \phi_0 = \xi$ and $\phi = \phi_0(1 + \xi/\phi_0)$. Hence, by expanding into series for two terms and collecting, we have

$$u = \left(\frac{\phi}{\phi_0}\right)^{3/2}\left\{R\cos\xi + S\sin\xi - R\sin\xi\,\frac{3}{2\phi_0} + R\xi\sin\xi\,\frac{15}{8\phi_0^2} - S\xi\cos\xi\,\frac{15}{8\phi_0^2} + S\sin\xi\,\frac{240}{128\phi_0^2}\right\} + L\phi^3\left(\frac{1}{\phi^4}-\frac{12}{\phi^6}\right).$$

It should next be noted that the coefficients R and S consist of the sum of terms of different orders of magnitude. In fact $u_0 - L/\phi_0^2$ may be considered as of the second order whereas $12L/\phi_0^4$ is of the fourth. So also in S we have u_0' of the second order (unless the orbit differs only infinitesimally from a circle, u' and u are of the same order) whereas $2L/\phi_0^3$ is of the third. If we rearrange the expression for u on this basis and at the same time expand $(\phi/\phi_0)^{3/2}$ we get as the final form.

$$\begin{aligned} u = {} & \left[\left(u_0 - \frac{L}{\phi_0^2}\right)\cos\xi + u_0'\sin\xi + \frac{L}{\phi_0^2}\right] \\ & + \frac{1}{\phi_0}\left[\frac{1}{2}\left(\frac{7L}{\phi_0^2}-3u_0\right)\sin\xi - \frac{2L}{\phi_0^2}\xi + \frac{3}{2}u_0'\xi\sin\xi + \frac{3}{2}\left(u_0-\frac{L}{\phi_0^2}\right)\xi\cos\xi\right] \\ & + \frac{1}{\phi_0^2}\left[\frac{15}{8}u_0'\sin\xi - \frac{24L}{\phi_0^2}\sin^2\tfrac{1}{2}\xi - \frac{15}{4}u_0'\xi\cos\xi - \frac{3}{8}\left(u_0-\frac{9L}{\phi_0^2}\right)\xi\sin\xi \right. \\ & \left. + \frac{3L}{\phi_0^2}\xi^2 + \frac{3}{8}\left(u_0'\sin\xi + \left(u_0-\frac{L}{\phi_0^2}\right)\cos\xi\right)\xi^2\right], \end{aligned} \tag{12}$$

and hence, on differentiating,

$$
\begin{aligned}
u' = &\left[u_0' \cos\xi - \left(u_0 - \frac{L}{\phi_0^2}\right)\sin\xi\right] \\
&+ \frac{1}{\phi_0}\left[\frac{3}{2}u_0'\left(\sin\xi + \xi\cos\xi\right) - \frac{3}{2}\left(u_0 - \frac{L}{\phi_0^2}\right)\xi\sin\xi - \frac{4L}{\phi_0^2}\sin^2\tfrac{1}{2}\xi\right] \\
&+ \frac{1}{\phi_0^2}\left[\frac{6L}{\phi_0^2}\,\xi - \frac{3}{8}\,u_0\sin\xi - \frac{69}{8}\,\frac{L}{\phi_0^2}\sin\xi + \frac{21}{8}\,u_0'\,\xi\sin\xi\right. \\
&\left. + \frac{3}{8}\left(u_0 + \frac{7L}{\phi_0^2}\right)\xi\cos\xi + \frac{3}{8}\left(u_0'\cos\xi - \left(u_0 - \frac{L}{\phi_0^2}\right)\sin\xi\right)\xi^2\right],
\end{aligned}
\tag{13}
$$

where the first, second, and third brackets denote respectively the first, second, and third approximations. The integration of the problem, both in exact and in approximate form, may therefore be considered as accomplished.

3. Discussion of the characteristics of the motion. In the first place let it be assumed that at the start the areal velocity is positive. Equation (3) shows that as the motion goes on and θ increases from its initial value 0, the areal velocity will steadily decrease and hence by equation (4) the argument ϕ will decrease. If we allow ϕ to approach zero in the complete integral (7), it appears that u is becoming infinite: for the terms

$$\phi^2 J_2(\phi), \quad \phi^2 Y_2(\phi), \quad \phi^2 J_2(\phi)\int\frac{Y_2(\phi)}{\phi^3}\,d\phi$$

approach finite limits and the integral*

$$\int\frac{J_2(\phi)}{\phi^3}\,d\phi$$

diverges for $\phi = 0$. Hence the particle will reach the attracting center when $\phi = 0$. It should be noted, however, that in an actual case the motion will be stopped before the particle reaches the center by the impact of the particle on the surface of the attracting mass. For another reason it is also valueless to investigate what happens to the orbit for very small values of ϕ, that is, when the particle is near the attracting center. We have assumed in setting up our equations that the velocity of the particle did not become very great.

* This follows immediately on substituting the principal values of the functions obtainable from the developments in series. See Gray and Mathews, p. 11 and p. 14.

Hence the particle should not be allowed to approach the center too closely. Hence $3Vh/K$ is an upper limit for θ, and we may say that the number of revolutions that the particle can make is less than

$$N = \frac{3Vh}{2\pi K}. \tag{14}$$

This number for any given particle depends solely on the initial areal velocity and not on the velocity of projection or the initial distance from the center, except in so far as they conjointly determine the areal velocity.

The approximate solution (12) will serve to discuss the motion for a few dozen revolutions from the initial position $\phi = \phi_0$. If it be assumed that the polar axis is so chosen that $u_0' = 0$, that is, so that the particle starts from an apse, then the first approximation may be put in the form

$$u = \frac{L}{\phi_0^2}(1 + e\cos\xi), \qquad e = \frac{u_0 - \frac{L}{\phi_0^2}}{\frac{L}{\phi_0^2}}.$$

This is a conic section as might have been expected. In particular it is that conic section which the particle would described under the action of the gravitational attraction diminished by the action of the direct pressure of light and free from the Doppler effects. This conic I shall call the osculating conic of the orbit.

To find the amount by which the particle falls in from the apse toward the center during the first complete revolution it is merely necessary to subtract the values of the radius vector as given by (12) for $\xi = 0$ and $\xi = -2\pi$. Hence to the second approximation we have

$$\Delta u_0 = u_0 \frac{1}{N}\frac{1}{1+e}\left(2 - \frac{3}{2}e\right), \qquad \Delta r_0 = -r_0 \frac{1}{N}\frac{1}{1+e}\left(2 - \frac{3}{2}e\right).$$

This quantity Δr_0 is always negative, showing an approach to the center in all cases,* and varies between the limits $r_0/4N$ and $2r_0/N$ as the osculating conic varies from a parabola to a circle. It is to be noticed that, if the eccentricity is small, the particle falls in toward the center by an amount about twice as

* We have assumed tacitly that $L > 0$, that is, that the attraction is greater than the pressure of light.

great as that required to carry it to the center in N revolutions if it moved in by the same amount during each revolution. This need not be perplexing, for according to the formula the particle is always falling in by a certain percentage of the distance remaining (not of the original total distance). It therefore appears necessary that the particle should fall in by an amount which at any rate for orbits of small eccentricity is decreasing from revolution to revolution. In fact if the increment of the radius vector be found to another approximation, it appears that such is the case.*

As the eccentricity of the osculating ellipse seems to be of importance, it appears worth while to calculate the change of eccentricity during a revolution. Formula (13) shows that if the particle starts at an apse, it will be at an apse after a complete revolution provided we disregard the effect of the last approximation. This fact will materially simplify the discussion of the eccentricity. It enables us to write the equation of the osculating ellipse after one revolution as

$$u = \left(u_0 + \Delta u_0 - \frac{L}{(\phi_0 - 2\pi)^2}\right) \cos \xi + \frac{L}{(\phi_0 - 2\pi)^2}$$

On calculating the eccentricity of this ellipse by the usual formula we find

$$\Delta e = -\frac{7}{2}\frac{e}{N}. \tag{15}$$

Hence it appears that the ellipse is always losing eccentricity. To speak roughly one may say that the orbit is losing its eccentricity from about two to about fifteen times as fast as it loses its perihelion distance according as the orbit is approximately circular or approximately parabolic.

Equation (13) shows that although there is not much change in the angular position of the apse, there is some. Let us assume that the apse advances by the amount ϵ, so that if $\xi = 0$ is a root of u', then $\xi = -2\pi - \epsilon$ is the next but one. It has been seen that ϵ is small of the order $1/N^2$. The second term of the first approximation gives a term in ϵ. All the terms of the second

* If I understand Poynting aright he states in *Nature*, loc. cit. p. 92, second column, that the particle will fall in farther during the succeeding revolutions than during the preceding ones. As Poynting very seldom makes a mistake, I have given the matter a great deal of attention but fail to find any error in my work. Moreover Poynting, in his earlier paper, obtains a formula which is interpretable in much the same way as mine. Unfortunately I cannot find that he has printed any mathematical treatment of the subject since then.

approximation are too small to contribute anything. The first and fifth terms of the third approximation contribute quantities independent of ϵ. The equation for ϵ therefore is

$$\left(u_0 - \frac{L}{\phi_0^2}\right)\epsilon = \frac{1}{\phi_0^2}\left[\frac{9L}{\phi_0^2}\,2\pi + \frac{3}{8}\left(u_0 - \frac{L}{\phi_0^2}\right)2\pi\right].$$

Hence

$$\epsilon = \frac{1}{2\pi N^2}\,\frac{1}{e}\left(9 + \frac{3}{8}e\right) = \frac{360}{\phi_0^2}\,\frac{1}{e}\left(9 + \frac{3}{8}e\right) \text{ degrees},$$

and there is a slight advance of the apse in the direction of motion. If the eccentricity is very small, that is, small of an order approaching $1/N^2$ the apse may advance over a considerable angle during a revolution.*

So much for the discussion of what happens in the vicinity of the start. We are now in a position to take up more general considerations. From the behavior of the apse investigated above, it should appear that as ϕ decreases from its initial value ϕ_0, the apses draw further and further apart; and conversely if we follow the motion backward allowing ϕ to increase, the apses will follow each other at intervals diminishing toward the limit π. Now this is what we should obtain on general considerations applied to formula (11). For the apses correspond to roots of u', that is, to roots of the expression

$$CJ_1(\phi) + DZ_1(\phi) - \frac{2L}{\phi^5}.$$

Here C and D are definite constants. To be sure they are very small in the cases we have been considering, in fact of the order $\phi_0^{-7/2}$. Nevertheless for values of ϕ sufficiently large, the third term, $-2L/\phi^5$, may be neglected relatively to the first two. Then the roots of u' may be investigated by the usual methods employed in discussing the roots of Bessel functions,† and we shall thereby obtain precisely the result concerning the advance of the apse that we have already obtained from our computations. As the value of u' becomes infinite when $\phi = 0$, it is clear that u' must have a definite smallest root corresponding to the last apse which the orbit possesses, after which it coils in around the attracting center with an essentially spiral form. It is not impos-

* It is, however, by no means certain that the orbit would keep its approximately conical nature for such small values of e: it might already have become spiral. See infra.

† Gray and Mathews, chap. 5.

sible to get a very rough estimate of where this last root occurs. For there can be no roots for values of ϕ so small that the third term above is larger than the other two. If we compare merely the magnitudes of ϕ, ϕ_0, C, and D, we find that the condition is fulfilled when $\phi = \phi_0^{7/9}$. To get any more accurate estimation seems not to be worth while.

Similar considerations may be applied to formula (10) to discuss the roots of u. If the test suggested above be used, it appears that the roots of u are likely to be found for values of ϕ of the same order of magnitude as the initial value ϕ_0. That this is correct follows from the approximate formula for u. Here, if the eccentricity e is greater then unity, there are roots of u in the immediate neighborhood of ϕ_0. If e is considerably less than unity, the motion of the particle may be followed back through a very large number of revolutions before the eccentricity shall have increased to unity. Nevertheless the rate of increase of e as given by (15) is tolerably fast, and it would be safe* to say that the first root of u would appear before ϕ had increased to many times ϕ_0. Now whenever u has a root, the particle has retreated to infinity. Hence it follows that not only is the number of revolutions which a particle can make inward toward the center a finite number N, but *the total number of revolutions which a particle can make in infinite time about an attracting luminous center is finite.* This result is so different from that of the ordinary theory of attraction as to be very striking. In fact, it could hardly be credited if the analysis had not assured its correctness. For values of ϕ greater than the first root of u the physical interpretation is simply that the particle is describing a hyperbolic type of path. As ϕ gets larger and larger the roots of u become separated by intervals equal more and more nearly to π. In other words the path is becoming more and more nearly straight. The analytical deductions, therefore, are seen to fulfil in every respect the conditions anticipated by the physical intuition.

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* In fact if we express C and D in terms of e_0 and if we assume that the particle starts at an apse where ϕ_0 has such a value that $Z_2(\phi_0) = 0$, then $J_1(\phi_0) = 0$ and the equation for roots of u may be written roughly in the form

$$\frac{1}{\phi_0}\frac{L}{\phi_0^2}e_0 Z_1(\phi_0) J_1(\phi) = \frac{L}{\phi^4}, \qquad Z_1(\phi_0) = \left(\frac{\pi}{2\phi_0}\right)^{1/2}.$$

Hence we may look for roots of u in the neighborhood of

$$\phi^{7/2} = \frac{1}{e_0}\phi_0^{7/2}, \qquad \phi = e_0^{-2/7}\phi_0.$$

In closing we may take the time to obtain numerically some of the quantities which we have been discussing. The case of particular interest is that of a particle revolving about the sun. The following data may be assumed:

velocity of light, $V = 3 \times 10^{10}$
distance of sun from earth $= 493 \times V$
acceleration of earth in orbit $= 59 \times 10^{-2}$
pressure of sunlight at earth's distance $= 59 \times 10^{-6}$.

All these quantities are in the C. G. S. system. The constant C in equation (1) has the value

$$C = 59 \times 10^{-2} \times 493^{-2}\ V^2 = 1{\cdot}29 \times 10^{26}.$$

The constant K may be determined in terms of the radius a and the density ρ of the particle as

$$K\colon C = 59 \times 10^{-6}\ \pi a^2 : \tfrac{4}{3}\pi\rho a^3 59 \times 10^{-2}, \quad K = \tfrac{3}{4}\rho^{-1}a^{-1}10^{-4}\,C.$$

If the particle has a density about equal to that of the earth and a radius of about 0·01cm, then $K = 1{\cdot}76 \times 10^{23}$. It should be noted that K is increased by a decrease in the radius or density, and that the value of K here obtained is certainly small for such particles as meteoric dust finely divided. Suppose that the initial double areal velocity is 45×10^{18} which is roughly that of the earth. Then if a particle be started at an apse at the earth's distance from the sun,

$$\phi_0 = 2{\cdot}3 \times 10^{7}, \quad N = 3{\cdot}66 \times 10^{6}, \quad e = 0{\cdot}052.$$

Hence the distance of fall from an apse will be 80·6km during the first revolution. The change of eccentricity will be about one millionth of its value. The apse will advance about one half of one millionth of a second of arc. It is very doubtful whether the particle could make so many as 10^7 revolutions altogether. If the particle had a radius of one tenth the amount above assumed, the fall would be ten times as great and so would the diminution of the eccentricity: the rotation of the apse, however, would be one hundred times as great as it was before.

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